THE CAPON-MVDR ALGORITHM: THRESHOLD SNR PREDICTION AND THE PROBABILITY OF RESOLUTION

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ABSTRACT

The threshold region mean squared error (MSE) performance of the Capon-MVDR algorithm is predicted via an adaptation of an interval error based method referred to herein as the method of interval errors (MIE). MIE requires good approximations of two quantities: (i) interval error probabilities, and (ii) the algorithm asymptotic (SNR$\rightarrow\infty$) MSE performance. Exact pairwise error probabilities for the Capon (and Bartlett) algorithm are derived herein that include finite sample effects for an arbitrary colored data covariance; with the Union Bound, accurate approximations of the interval error probabilities are obtained. Further, with the large sample MSE predictions of Vaidyanathan and Buckley, MIE accurately predicts the signal-to-noise ratio (SNR) threshold point, below which the Capon algorithm MSE performance degrades swiftly. A two-point measure of the probability of resolution is defined for the Capon algorithm that accurately predicts the SNR at which sources of arbitrary closeness become resolvable.

1. INTRODUCTION

The threshold region mean squared error (MSE) performance of signal parameter estimates derived from the Capon high-resolution spectral estimator, *a.k.a* the minimum variance distortionless response (MVDR) spectral estimator, is the primary subject of this analysis. Similar to maximum-likelihood (ML) methods, the Capon processor is a beamscan type algorithm involving a nonlinear maximization of an objective search function (OSF). Parameter estimation algorithms requiring nonlinear searches typically exhibit a threshold effect in MSE performance. Below a specific signal-to-noise ratio (SNR) called the estimation threshold, the MSE departs from the asymptotic MSE performance and rises rapidly (see pp. 278–286 of [13]). Clearly, accurate prediction of this threshold SNR is of great practical significance for system design/analysis, particularly for methods capable of significant resolving power at SNRs too low for signal detection. Below the estimation threshold SNR, the MSE rises until it reaches a maximum that at times can be well approximated by the variance of an estimate that is assumed uniformly distributed over the search domain. The SNR at which the MSE performance achieves this level of futility is called the no information point. Figure 1 illustrates this composite MSE performance typical of nonlinear estimation schemes. This composite MSE behavior is typical of nonlinear ML estimation [13], but likewise occurs with the Capon spectral estimator [14]. Although well-known, accurate prediction of this composite performance curve for the Capon algorithm remains an open problem. The goal of this analysis is to predict this MSE curve for the Capon algorithm with primary emphasis on threshold region performance from which an accurate prediction of the threshold SNR can be obtained.

A classical method of MSE approximation, referred to herein as the method of interval errors (MIE), was introduced by Van Trees [13] and provides a means of predicting threshold region performance of nonlinear estimation techniques. Variants of MIE have been applied to subspace based methods and ML estimation techniques with much success [10, 6, 15, 1, 8]. MIE requires good approximations of two quantities: (i) interval error probabilities, and (ii) the asymptotic MSE performance. Both of these quantities are algorithm dependent. The interval error probabilities quantify the likelihood that the estimator derives its signal parameter estimate from a false peak of the ambiguity function as opposed to the true peak. These probabilities are approximated via the Union Bound in conjunction with exact pairwise error probabilities for the Capon estimator that are derived herein; these derived probabilities account for arbitrary colored data covariance structure as well as finite sample support training effects [3, 7]. These calculations naturally lead to a two-point measure of the Capon probability of resolution from which accurate prediction of the SNRs required to resolve closely spaced sources is possible.

2. THE CAPON METHOD

The Capon high-resolution algorithm is well-known [2, 3, 14] and its performance has been studied extensively. It will be assumed in this section that all sources are well separated (by at least a beamwidth or Rayleigh distance) and possess SNRs that exceed the estimation threshold. In addition it is assumed that signals are mutually incoherent (coherent sources are not resolvable with the Capon algorithm), and that the total number of signals present in the data is known.

2.1. Capon's Approach

Given a set of independent identically distributed signal bearing observations $\mathbf{X} = [\mathbf{x}(1)|\mathbf{x}(2)|\cdots|\mathbf{x}(L)]$ where each vector is

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$N \times 1$ complex circular Gaussian, *i.e.* $\mathbf{x}(l) \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R})$, $l = 1, 2, \ldots, L$, Capon proposed the following power spectral estimator:

$$P_{Capon}(\theta) = \frac{1}{L-N} \cdot \frac{1}{\mathbf{v}^H(\theta)\widehat{\mathbf{R}}^{-1}\mathbf{v}(\theta)} \tag{1}$$

where $\mathbf{v}(\theta)$ is the assumed array response, $\widehat{\mathbf{R}} = \mathbf{X}\mathbf{X}^H$, and $\mathbf{R} = \mathbf{R}_N + \sigma_S^2 \cdot \mathbf{v}(\theta_T)\mathbf{v}^H(\theta_T)$, where $\mathbf{R}_N$ is background noise (possibly colored, but absent of signal-like interference). The maximum output provides an estimate of the signal power σ_S^2 and the signal parameter estimate is given by the scan value of θ that achieves this maximum; namely,

$$\widehat{\theta} = \arg\max_{\theta} P_{Capon}(\theta) \tag{2}$$

(assuming a single signal is present). It shall be assumed that K signals are present in the data, and that the Capon parameter estimates $\widehat{\theta}_k$, $k = 1, 2, \ldots, K$, are obtained as the arguments of the K largest peaks of $P_{Capon}(\theta)$.

2.2. Large Sample MSE of the Capon Algorithm

The large sample ($L \gg N$) local error MSE performance of the Capon signal parameter estimator has been theoretically analyzed by several authors. Stoica et. al. [11], Vaidyanathan and Buckley (VB) [12], and Hawkes and Nehorai [5] exploit Taylor's theorem and complex gradient methods to approximate the MSE. VB provide an additional bias term via a second order Taylor series expansion that is particularly useful for capturing finite sample effects and a broader range of values for L and SNR. The results of VB will be used herein, and the local error MSE approximation obtained thereby shall be denoted by the symbol $\sigma_{VB}^2(\theta_k)$.

3. THRESHOLD REGION MSE PREDICTION

This section describes the method of interval errors (MIE) for MSE prediction and its adaptation to the Capon algorithm. The reader is also referred to [1, 15] for an excellent description of MIE in the context of ML estimation.

3.1. Method of Interval Errors

MIE builds upon the two regions of the composite MSE curve of Figure 1 that are given by the asymptotes of the SNR; namely, the no information (SNR$\rightarrow$ 0) and asymptotic (SNR$\rightarrow \infty$) regions. Define the conditioning event

$$\mathcal{A} = \{\text{True source parameters are } \theta_k, k = 1, 2, \ldots, K\}. \tag{3}$$

MIE decomposes the MSE expression into two components: "no interval errors" (NIE), and "interval errors" (IE)

$$\begin{aligned} E\left\{\left(\widehat{\theta}_k - \theta_k\right)^2 \middle| \mathcal{A}\right\} &= \int p_{\widehat{\theta}_k}\left(\widehat{\theta}_k = \theta_0 \middle| \mathcal{A}\right)(\theta_0 - \theta_k)^2 \, d\theta_0 \\ &= \Pr(\text{NIE} \mid \mathcal{A})\, E\left\{\left(\widehat{\theta}_k - \theta_k\right)^2 \middle| \text{NIE}, \mathcal{A}\right\} + \\ &\quad \Pr(\text{IE} \mid \mathcal{A})\, E\left\{\left(\widehat{\theta}_k - \theta_k\right)^2 \middle| \text{IE}, \mathcal{A}\right\} \end{aligned}$$

(see equation (127) on p. 282 of [13]). The parameter search space, *i.e.* the scanning domain for θ, is divided into disjoint mutually exclusive intervals based on the characteristics of underlying ambiguity function $\psi_{Capon}(\theta) \triangleq \frac{1}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}$, that depends on $\mathbf{R}$, and hence is a function of the K SNRs of the K signals present.

3.1.1. *Multiple Sources: $K \geq 1$*

Assume arbitrary $K \geq 1$; in addition assume that these K signals are well separated by at least a beamwidth (thus, negligible likelihood of intersource errors). The extension of MIE to multiple sources is accomplished by expanding the "no interval errors" set to include all local neighborhoods of the K peaks in the ambiguity function due to the K sources present (clearly, all other intervals lead to IE). *The large sample MSE approximation obtained via $\sigma_{VB}^2(\theta_k)$ will be used to describe the "no interval errors" component contribution to the over MSE of the k-th source parameter Capon estimate.*

Let all local maxima within the signal parameter domain of interest of the ambiguity function when evaluated at K large SNRs (large enough that the ambiguity function has a local maximum at every true parameter value θ_k) be given by the finite set $\mathcal{M} = \{\theta \mid \theta_1, \theta_2, \ldots, \theta_{K+M-1}\}$ where θ_k for $k = 1, 2, \ldots, K$ represent the peaks due to the K sources, and θ_k for $k = K+1, K+2, \ldots, K+M-1$ represent all other non-source local maxima.[1] The total MSE for this Capon parameter estimate can be approximated by

$$\begin{aligned} &E\left\{\left(\widehat{\theta}_k - \theta_k\right)^2 \middle| \mathcal{A}\right\} \simeq \\ &\left[1 - \sum_{m=K+1}^{K+M-1} p\left(\widehat{\theta}_k = \theta_m \middle| \mathcal{A}\right)\right] \cdot \sigma_{VB}^2(\theta_k) \\ &\quad + \sum_{m=K+1}^{K+M-1} p\left(\widehat{\theta}_k = \theta_m \middle| \mathcal{A}\right)(\theta_m - \theta_k)^2 . \end{aligned} \tag{4}$$

The interval error probability $p\left(\widehat{\theta}_k = \theta_m \middle| \mathcal{A}\right)$ represents the likelihood of the Capon search algorithm choosing a value associated with the false peak located at $\theta = \theta_m$ as an estimate for θ_k, when the K true signals are located at parameter values $\theta = \theta_k$, $k = 1, 2, \ldots, K$.

As in [1, 15], the dominant term of the Union Bound (UB) can be used to approximate the interval error probabilities:

$$\begin{aligned} &p\left(\widehat{\theta}_k = \theta_m \middle| \mathcal{A}\right) \simeq \\ &\Pr\left[P_{Capon}(\theta_m) > P_{Capon}(\theta_k) \middle| \mathcal{A}\right]. \end{aligned} \tag{5}$$

This modified UB approximation is remarkably accurate in the vicinity of the estimation threshold SNR, but tends to over predict the MSE in the no information region. *Thus, the minimum of (4) and the worse case MSE obtained with an estimate $\widehat{\theta}_k$ that is uniformly distributed over the parameter search space will be chosen as the MSE prediction.*

3.2. Capon Pairwise Error Probabilities

The desired pairwise error probabilities are of the form

$$P_e^{Capon}(\theta_a|\theta_b) \triangleq \Pr\left[P_{Capon}(\theta_a) > P_{Capon}(\theta_b) \middle| \mathcal{A}\right]. \tag{6}$$

Define the following function

$$\mathcal{F}(x, N_0) \triangleq \frac{x^{N_0}}{(1+x)^{2N_0-1}} \sum_{k=0}^{N_0-1} \binom{2N_0-1}{k+N_0} \cdot x^k \tag{7}$$

[1] Such SNRs will exist provided that no array response mismatch is present, *i.e.* provided that the array responses used to compute $P_{Capon}(\theta)$ match the K array responses existing in the true data covariance $\mathbf{R}$ for θ_k, $k = 1, 2, \ldots, K$.

where $\mathcal{F}(x, N_0)$ is the cumulative distribution function for a special case of the complex central F statistic. The algorithm for computing the pairwise error probabilities for the Capon estimator is as follows:

1. Define the $N \times 2$ matrix $\mathbf{V} = [\mathbf{v}(\theta_a)|\mathbf{v}(\theta_b)]$ and choose the desired covariance parameter $\mathbf{R}$.
2. Perform the following QR-decomposition

$$\mathbf{R}^{-1/2}\mathbf{V} = \mathbf{Q}^H \left[\begin{array}{c} \mathbf{\Delta}_{2\times 2} \\ \mathbf{0}_{(N-2)\times 2} \end{array} \right]; \text{let } \mathbf{\Delta} = [\boldsymbol{\delta}_1|\boldsymbol{\delta}_2]. \quad (8)$$

3. Define the matrix $\boldsymbol{\delta}_2\boldsymbol{\delta}_2^H + F \cdot \boldsymbol{\delta}_1\boldsymbol{\delta}_1^H$ for any non-positive real number $F \leq 0$, and its two eigenvalues as $\lambda_1(F)$ and $\lambda_2(F)$, and their ratio as $l_\lambda(F) = -\lambda_2(F)/\lambda_1(F)$.
4. The desired exact pairwise error probability for the Capon algorithm is given by the expression

$$\boxed{\begin{array}{c} P_e^{Capon}(\theta_a|\theta_b) = 0.5 \cdot \{1 + \text{sign}\,[\lambda_1(-1)]\} \\ -\text{sign}[\lambda_1(-1)] \cdot \mathcal{F}[l_\lambda(-1), L - N + 2]. \end{array}} \quad (9)$$

See [9] for derivation.

4. THE CAPON PROBABILITY OF RESOLUTION

A useful measure of the probability of resolution can be defined that provides excellent prediction of the SNR at which sources can be resolved by the Capon algorithm. For a two closely spaced sources scenario of the form $\mathbf{R} = \mathbf{R}_N + \sigma_{S_a}^2\mathbf{v}(\theta_0)\mathbf{v}^H(\theta_0) + \sigma_{S_b}^2\mathbf{v}(\theta_0 + \delta\theta)\mathbf{v}^H(\theta_0 + \delta\theta)$, define parameter θ_{MP} as the parameter value of the source with the smallest power out of the ambiguity function, *i.e.* $\theta_{MP} \triangleq \arg \min_{\theta_0, \theta_0+\delta\theta} \psi_{Capon}(\theta)$. A two point measure of the probability of resolution can be defined as

$$P_{res}^{Capon}(\theta_0, \theta_0 + \delta\theta) \triangleq \Pr\left[P_{Capon}\left(\theta_0 + \frac{\delta\theta}{2}\right) \leq \rho \cdot P_{Capon}(\theta_{MP})\right] \quad (10)$$

where $0 \leq \rho \leq 1$. The parameter ρ essentially defines the desired "dip" in Capon output power between two closely spaced sources. Similar measures of resolution have been proposed [4, 14]. The algorithm for computing the Capon two point probability of resolution is the same as that for $P_e^{Capon}(\theta_a|\theta_b)$ with $\theta_a = \theta_{MP}$, $\theta_b = \theta_0 + \delta\theta/2$, and $F = -1/\rho$. A discussion of performance with closely spaced sources utilizing this measure is given in [9].

5. NUMERICAL EXAMPLES

Consider a Direction of Arrival (DOA) estimation scenario involving a single source and a set of signal bearing snapshots $\mathbf{x}(l) \sim \mathcal{CN}[\mathbf{0}, \mathbf{I} + \sigma_S^2\mathbf{v}(\theta_T)\mathbf{v}^H(\theta_T)]$, $l = 1, 2, \ldots, L$, for an $N = 18$ element uniform linear array (ULA) with slightly less than $\lambda/2$ element spacing. The array has a 3dB beamwidth of 7.2 degrees and the desired target signal is arbitrarily placed at $\theta_T = 90$ degrees (array broadside). The signal parameter search space of interest is defined to be $\theta \in [60^\circ, 120^\circ]$. The signal parameter to be estimated is simply the scalar angle of arrival $\theta = \theta_T$. Figure 2 illustrates the Monte Carlo based MSE performance of the Capon algorithm alongside its MIE prediction and the Cramér-Rao Bound (CRB) for sample support cases $L = 1.5N, 2N$ and $3N$ snapshots. The Capon estimator clearly is not asymptotically (L fixed, SNR$\rightarrow \infty$) efficient, since increasing the SNR does not bring its MSE performance closer to the CRB, hence the need for analyses such as [11, 12, 5]. The VB MSE prediction is plotted for the $L = 2N$ case to illustrate that this large sample Taylor Series based approximation is a local one and is only valid above the estimation threshold SNR. The goal of MIE is to reasonably predict MSE performance well into the estimation threshold region. Note from the $L = 2N$ case in Figure 2 that MIE continues with accurate prediction well into the threshold region by accounting for global errors, whereas the VB MSE prediction becomes inaccurate. For example, the VB prediction is off by about 5dB for the SNR required for a 6 to 1 beam split ratio (RMSE$\simeq -7.5$dB). Note that in the no information region the UB approximation begins to over predict the MSE. Thus, it is allowed to increase until it maxes at the MSE obtained for an estimate that is uniformly distributed over the signal parameter domain of interest.

Next consider the same scenario, but with an additional source of equal power included in the environment at 70 degrees. The MSE performance of both signal parameter estimates is illustrated in Figures 3–4. The MIE predictions remain quite accurate.

6. CONCLUSIONS

The method of interval errors (MIE) has been successfully adapted and extended to the Capon-MVDR algorithm, providing remarkably accurate prediction of the MSE threshold SNRs for an arbitrary number of well separated sources. These SNRs are predicted via simple finite sum expressions for the pairwise error probabilities, involving no numerical integration, and circumventing the need for many time consuming and cumbersome Monte Carlo simulations. A new two-point measure of the Capon probability of resolution was proposed that accurately predicts the SNRs necessary for mutual source resolvability for sources of arbitrary closeness. Both represent valuable system design/analysis tools.

7. REFERENCES

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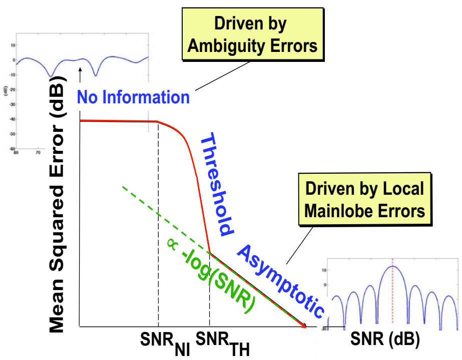

Fig. 1. Composite MSE Curve for Parameter Estimation

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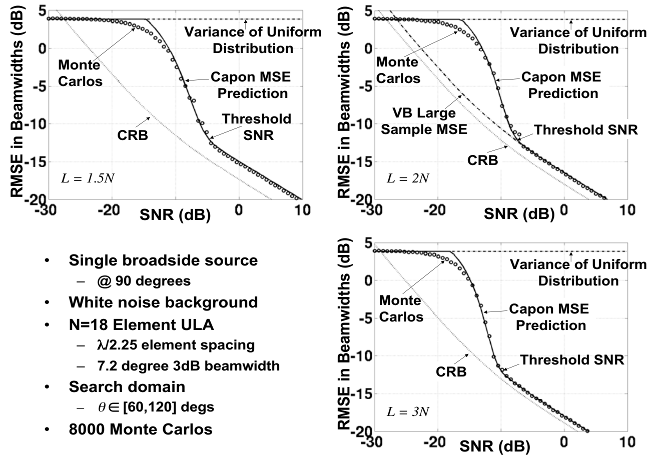

Fig. 2. Single Source Capon MSE Performance, $\theta_T = 90°$, $L = 1.5N, 2N, 3N$

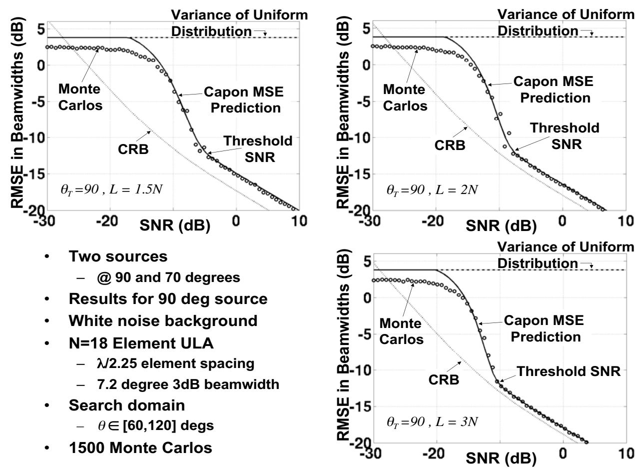

Fig. 3. Two Source Capon MSE Performance, $\theta_1 = 90°$, $\theta_2 = 70°$, $L = 1.5N, 2N, 3N$

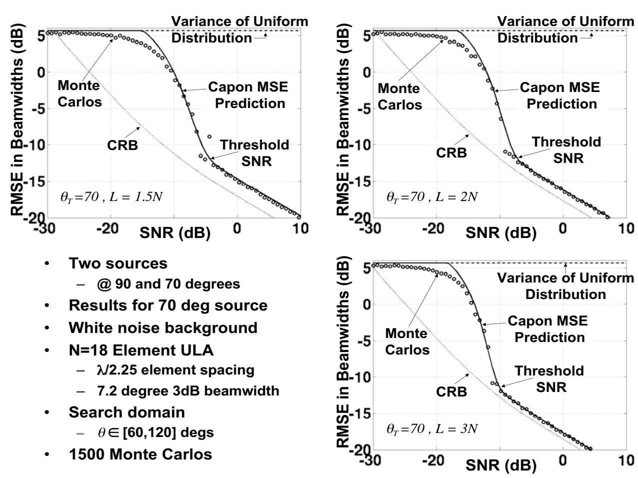

Fig. 4. Two Source Capon MSE Performance, $\theta_1 = 90°$, $\theta_2 = 70°$, $L = 1.5N, 2N, 3N$