SPATIALLY CODED SIGNAL MODEL FOR ACTIVE ARRAYS

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ABSTRACT

This paper addresses the problem of target detection and localization by radar or active sonar systems. A novel configuration in which the transmitted signals are spatially coded, is proposed. The main advantages of this new configuration are: avoid beam-shape loss, larger virtual array aperture and therefore narrower beams, increase the angular resolution, and ability to detect and localize greater number of targets. This configuration enables array processing in the transmit mode in addition to the receive mode. The generalized likelihood ratio test (GLRT) and the maximum-likelihood (ML) estimator are derived for target detection and localization according to the new model configuration. The performance of the array processing algorithms for this problem is studied theoretically and via simulations.

1. INTRODUCTION

In conventional active systems for target detection and localization, such as radars or active sonars, a directional signal is usually transmitted, and the target echo signal is processed. In the recent years many array processing techniques have been developed for target detection and localization in the receive mode. In the transmit mode, when a single signal is transmitted by the array, no array processing technique can be implemented. Array processing in the transmit mode could be possible only when the transmitted signal is coded in space, that is, different elements of the array transmit different signals.

Transmission of orthogonal signals from an array is commonly used in communication systems [1]. Passive localization of orthogonal signals with known waveforms was investigated in [2]. In [3] it is shown that the conventional configuration of one transmitter and two receivers and an alternative configuration of two transmitters and one receiver, are equivalent in terms of Cramér-Rao Bound (CRB) on bearing estimation. This configuration requires radiating two uncorrelated signals from two transmitters. The potential advantage of this configuration over the conventional one is in applications where the receiving elements are to be placed on a platform of limited size. The results in [3] are extended to the case of transducers in [4]. Three possible combinations of four transducers: 1) one transmitter and three receivers, 2) two transmitters and two receivers, and 3) three transmitters and one receiver were investigated. These configurations have identical performance in terms of angle estimation accuracy, where the transmitting signals are orthogonal.

In this paper, we present a configuration for spatial coding of the transmitted signal, and analyze its properties in target detection and localization. In the new configuration, each element or sub-array of elements transmits a different signal. In the receive mode, the output of each element or subarray is matched to all the transmitted signals. Array processing algorithms are applied to the output of the matched-filters, based on the transmit-receive model. The main advantages of this new configuration are: 1) avoid beam-shape loss, 2) larger virtual array aperture, and therefore narrower beams, 3) increase the angular resolution, and 4) ability to detect and localize a larger number of targets.

2. SPATIALLY CODED SIGNAL MODEL

Consider an M element antenna array transmitting M narrowband signals. The N samples of baseband equivalent signals are denoted by $\{\mathbf{s}[n]\}_{n=1}^{N}$ with correlation matrix

$$\mathbf{R_s} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{s}[n]\mathbf{s}^H[n] = \begin{bmatrix} 1 & \beta_{12} & \cdots & \beta_{1M} \\ \beta_{21} & 1 & \cdots & \beta_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{M1} & \beta_{M2} & \cdots & 1 \end{bmatrix}, \tag{1}$$

where β_{ij} is the correlation coefficient between the ith and jth signals, and the phases of $\{\beta_{ij}\}_{i,j=1}^{M}$ control the transmitted beam direction. In the case of coherent transmitted signals, the rank of the matrix $\mathbf{R_s}$ is reduced. In common radar systems, for example, coherent signals are transmitted from the array and therefore the rank of $\mathbf{R_s}$ is equal to one[1].

In the presence of a single target at direction θ, the received signal at the mth element of the array located at $(x_m^{(1)}, x_m^{(2)})$ (see Fig. 1), is given by:

$$y_m[n] = \alpha \sum_{i=1}^{M} A_{mi}(\theta)s_i[n] + w_m[n] \quad m = 1,\ldots,M, \; n = 1,\ldots,N \tag{2}$$

where α is the complex amplitude of the reflected signal from the target, $w_m[n]$ is the additive noise at element m, and $A_{mi}(\theta) = exp(-jw_c\tau_{mi})$ describes the total phase delay of the signal, transmitted by the mth element and received by the ith element, where w_c is the carrier frequency. The total delay from the ith transmitting element to the mth receiving element for the far-field case is $\tau_{mi} = \tau_m + \tau_i$,

[1]The different elements transmit the same signal with phase shifts for beam steering.

where $\tau_i = \frac{x_i^{(1)} sin(\theta) + x_i^{(2)} cos(\theta)}{c}$ and c is the propagation speed. The array response $A_{mi}(\theta)$ can be decomposed as: $A_{mi}(\theta) = a_m(\theta) a_i(\theta)$, where $a_m(\theta) = exp(-jw_c\tau_m)$.

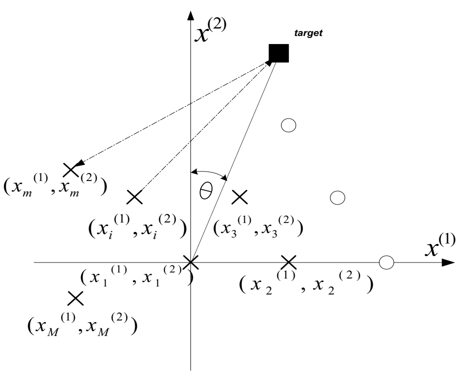

Figure 1: Array configuration

In matrix notation, Eq. (2) can be written as:

$$\mathbf{y}[n] = \alpha \mathbf{A}(\theta)\mathbf{s}[n] + \mathbf{w}[n] \quad n = 1, \ldots, N \ , \tag{3}$$

where $\mathbf{A}(\theta) = \mathbf{a}(\theta)\mathbf{a}^T(\theta)$ is the matrix of array response, $\mathbf{y}[n]$, $\mathbf{s}[n]$ and $\mathbf{w}[n]$ are the vectors of received signal, transmitted signal and additive noise, respectively.

The vector of unknown parameters includes the DOA and the complex amplitude: $\boldsymbol{\xi} = [\theta \ \alpha]^T$, which is modeled as deterministic unknown. The noise vectors $\{\mathbf{w}[n]\}_{n=1}^N$ are assumed to be white, zero-mean, complex Gaussian with covariance matrix $\sigma_w^2 \mathbf{I}_M$, where $\mathbf{I}_M$ is an identity matrix of size M.

In this paper, we consider the single target case. Extension to multiple case is straightforward. The objective herein is to:

1. Derive the ML estimator for target DOA estimation based on the measurement model of (3);
2. Derive the GLRT for target detection from the measurements $\{\mathbf{y}[n]\}_{n=1}^N$ according to the following hypotheses:

$$\begin{aligned} H_0 : \mathbf{y}[n] = & \quad \mathbf{w}[n] \\ H_1 : \mathbf{y}[n] = & \quad \alpha \mathbf{A}(\theta)\mathbf{s}[n] + \mathbf{w}[n] \ ; \end{aligned} \tag{4}$$

3. Derivation of the CRB for target localization.

3. TARGET DETECTION AND DOA ESTIMATION

In this section we investigate the properties of the proposed model. First, the sufficient statistics for detection and estimation algorithms will be derived. In subsections 3.2 and 3.3, the ML and GLRT for DOA estimation and target detection are presented.

3.1. Sufficient Statistics

According to the assumptions stated in the previous section, the measurements are independent complex Gaussian vectors with $\mathbf{y}[n] \sim N^c(\alpha \mathbf{A}(\theta)\mathbf{s}[n], \sigma_w^2 \mathbf{I}_M)$. Hence, the log-likelihood function for estimating $\boldsymbol{\xi}$ from the data $\mathbf{Y} = (\mathbf{y}[1], \ldots, \mathbf{y}[N])$ can be expressed as

$$\log f_{\mathbf{Y}}(\mathbf{Y};\boldsymbol{\xi}) = \underbrace{-MN\log(\pi\sigma_w^2) - \frac{1}{\sigma_w^2}\sum_{n=1}^N \mathbf{y}^H[n]\mathbf{y}[n]}_{h(\mathbf{Y})}$$
$$\underbrace{+\frac{2}{\sigma_w^2} Re\left\{\alpha^* \sum_{n=1}^N \mathbf{s}^H[n]\mathbf{A}^H(\theta)\mathbf{y}[n]\right\}}_{g_1(\mathbf{Y},\boldsymbol{\xi})} \underbrace{-\frac{|\alpha|^2}{\sigma_w^2}\sum_{n=1}^N ||\mathbf{A}(\theta)\mathbf{s}[n]||^2}_{g_2(\boldsymbol{\xi})} .$$

We are interested in finding the sufficient statistics for estimating $\boldsymbol{\xi}$. According to the Neyman-Fisher factorization, the function $g_1(\cdot,\cdot)$ depends on the data $\mathbf{Y}$ only through the sufficient statistics. Denoting the columns of $\mathbf{A}(\theta)$ by $\{\mathbf{a}_m(\theta)\}_{m=1}^M$, we obtain:

$$g_1(\mathbf{Y},\boldsymbol{\xi}) = \frac{2}{\sigma_w^2} Re\left\{\alpha^* \sum_{m=1}^M \mathbf{a}_m^H(\theta)\boldsymbol{\eta}_m(\mathbf{Y})\right\} \tag{5}$$

where $\boldsymbol{\eta}_m(\mathbf{Y}) = \frac{1}{N}\sum_{n=1}^N \mathbf{y}[n]s_m^*[n]$ is the mth sufficient statistics and it is obtained by matching the observed data to the mth signal, $\{s_m[n]\}_{n=1}^N$. Note that

$$\begin{aligned} g_2(\boldsymbol{\xi}) = & -\frac{|\alpha|^2}{\sigma_w^2}\sum_{n=1}^N \mathbf{s}^H[n]\mathbf{A}^H(\theta)\mathbf{A}(\theta)\mathbf{s}[n] = \\ & -\frac{N|\alpha|^2}{\sigma_w^2} tr(\mathbf{A}^H(\theta)\mathbf{R_s}\mathbf{A}(\theta)) = -\frac{NM|\alpha|^2}{\sigma_w^2}\mathbf{a}^H(\theta)\mathbf{R_s}\mathbf{a}(\theta) \end{aligned} \tag{6}$$

Hence, the log-likelihood function of $\mathbf{Y}$ can be written as

$$\begin{aligned} \log f_{\mathbf{Y}}(\mathbf{Y};\boldsymbol{\xi}) = \quad & h(\mathbf{Y}) + \frac{2}{\sigma_w^2} Re\left\{N\alpha^* \sum_{m=1}^M \mathbf{a}_m^H(\theta)\boldsymbol{\eta}_m(\mathbf{Y})\right\} \\ & -\frac{NM|\alpha|^2}{\sigma_w^2}\mathbf{a}^H(\theta)\mathbf{R_s}\mathbf{a}(\theta) \ . \end{aligned} \tag{7}$$

It can be shown that for non-orthogonal signals, the sufficient statistics $\{\boldsymbol{\eta}_m(\mathbf{Y})\}_{m=1}^M$ are statistically dependent. For simplicity of the algorithms we are interested in independent sufficient statistics $\{\tilde{\boldsymbol{\eta}}_m(\mathbf{Y})\}_{m=1}^M$, which can be obtained as follows. The signal correlation matrix from (1) can be decomposed using SVD as $\mathbf{R_s} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H$, where $\mathbf{U}$ and $\boldsymbol{\Lambda}$ are the matrices of eigenvectors and eigenvalues of $\mathbf{R_s}$, respectively. Accordingly, the vector of independent signals can be obtained by

$$\tilde{\mathbf{s}}[n] = \boldsymbol{\Lambda}^{-1/2}\mathbf{U}^H\mathbf{s}[n] \ . \tag{8}$$

The modified sufficient statistics vector ia defined as

$$\boldsymbol{\eta} \triangleq vec[\tilde{\boldsymbol{\eta}}_1(\mathbf{Y}), \ldots, \tilde{\boldsymbol{\eta}}_M(\mathbf{Y})] = vec\left(\frac{1}{N}\sum_{n=1}^N \mathbf{y}[n]\tilde{\mathbf{s}}^H[n]\right) . \tag{9}$$

The configuration for obtaining the sufficient statistics from the data is described in Fig. 2. By inserting Eqs. (3) and (8) into (9), we obtain

$$\tilde{\boldsymbol{\eta}} = vec\left(\alpha\mathbf{A}(\theta)\frac{1}{N}\sum_{n=1}^N \mathbf{s}[n]\tilde{\mathbf{s}}^H[n] + \frac{1}{N}\sum_{n=1}^N \mathbf{w}[n]\tilde{\mathbf{s}}^H[n]\right). \tag{10}$$

Finally, Eq. (10) can be written in the form

$$\boldsymbol{\eta} = \alpha \mathbf{d}(\theta) + \mathbf{v} \ , \tag{11}$$

where $\mathbf{d}(\theta) = vec(\mathbf{A}(\theta)\mathbf{U}\boldsymbol{\Lambda}^{1/2})$ is the equivalent array response and $\mathbf{v} = vec\left(\sum_{n=1}^{N} \mathbf{w}[n]\tilde{\mathbf{s}}[n]^H\right) = [\mathbf{v}_1^T, \ldots, \mathbf{v}_M^T]^T$, whereas the $\{\mathbf{v}_m\}_{m=1}^M$ are i.i.d. with $\mathbf{v}_m \sim N^c(\mathbf{0}, \sigma_w^2 \mathbf{I}_M)$.

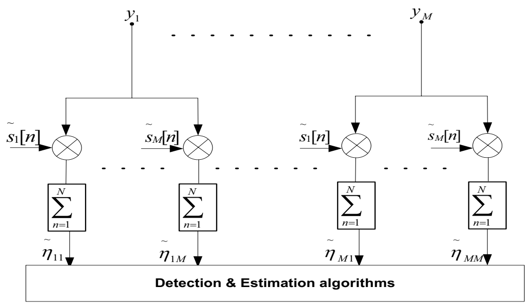

Figure 2: Sufficient statistics extraction

In order to illustrate the advantage of the proposed configuration (11), we examine an example with three array elements (M=3), which are located at vertexes of an equilateral triangle (see the first 3 x-points in Fig. 1). For incoherent signals, the passive equivalent array response can be written as $\mathbf{d}(\theta) = vec(\mathbf{A}(\theta))$ or in terms of phase delay we obtain

$$d_{m+i}(\theta) = A_{mi}(\theta) = exp(-jw_c\tau_{mi}), \ m, i = 1, \ldots, M. \tag{12}$$

By examining Eq. (12), we obtain 6 different delays (instead of 3 in the case of coherent signals), which are equivalent to 6 sensors: 3 original sensors and 3 virtual sensors which are located according to Fig. 1 (see o-points). Generally, the extended array geometry can be obtained by vector summation of all couples of the actual sensor locations. This phenomenon enables to double the virtual aperture of the array. This virtual aperture extension is very important for obtaining higher DOA estimation accuracy, narrower beams and therefore higher angular resolution, and better detection performance.

3.2. Maximum Likelihood (ML) estimation

Herein, the ML estimator for target localization for model (11) is given by

$$(\hat{\theta}, \hat{\alpha})_{ML} = arg \min_{\theta, \alpha} ||\boldsymbol{\eta} - \alpha \mathbf{d}(\theta)||^2 \ . \tag{13}$$

Differentiating (13) with respect to α and then equating to zero, we obtain

$$\hat{\alpha}_{ML} = (\mathbf{d}(\theta)^H \mathbf{d}(\theta))^{-1} \mathbf{d}(\theta) \boldsymbol{\eta} \tag{14}$$

and the ML estimator for θ is given by

$$\hat{\theta}_{ML} = arg \max_{\theta} \left\{ \boldsymbol{\eta}^H \mathbf{P}_d(\theta) \boldsymbol{\eta} \right\} \ . \tag{15}$$

where $\mathbf{P}_d(\theta) = \mathbf{d}(\theta)(\mathbf{d}(\theta)^H \mathbf{d}(\theta))^{-1} \mathbf{d}^H(\theta)$ is a projection matrix into the subspace spanned by $\mathbf{d}(\theta)$.

3.3. Detection

The GLRT for the two hypotheses defined in (4), is given by

$$L(\mathbf{Y}) = \log f_{\mathbf{Y}}(\mathbf{Y}; \hat{\theta}, \hat{\alpha}, H_1) - \log f_{\mathbf{Y}}(\mathbf{Y}; H_0) \underset{H_0}{\overset{H_1}{\gtrless}} \delta \ . \tag{16}$$

It can be shown that $\log f_{\mathbf{Y}}(\mathbf{Y}; H_0) = h(\mathbf{Y})$. Using Eqs. (7) and (14), the log-likelihood ratio function can be rewritten as

$$L(\mathbf{Y}) = \boldsymbol{\eta}^H \mathbf{P}_d(\hat{\theta}_{ML}) \boldsymbol{\eta} \ . \tag{17}$$

The threshold δ is set according to the desired false alarm rate. The statistics of $L(\mathbf{Y})$ under the two hypotheses are given by: $2L(\mathbf{Y}) \sim \begin{cases} \chi_2^2, & H_0 \\ {\chi'}_2^2(\rho), & H_1 \end{cases}$ where χ_2^2 and ${\chi'}_2^2(\rho)$ - central and non-central chi-squared distributions with 2 degrees of freedom, respectively; and ρ is the non-centrality parameter, which is equal to $\lambda = 2|\alpha|^2 \mathbf{d}(\theta)^H \mathbf{d}(\theta)$ (for orthogonal signals $\lambda = M$).

4. CRAMÉR-RAO BOUND

The CRB for estimation of θ according to the model in (3) is (see [5])

$$CRB(\theta) = \frac{tr(\mathbf{A}^H \mathbf{R_s} \mathbf{A})}{2SNR(tr(\dot{\mathbf{A}}^H \mathbf{R_s} \dot{\mathbf{A}}) tr(\mathbf{A}^H \mathbf{R_s} \mathbf{A}) - |tr(\dot{\mathbf{A}}^H \mathbf{R_s} \mathbf{A})|^2)} \tag{18}$$

where $SNR = \frac{|\alpha|^2}{\sigma_w^2}$, and $\dot{\mathbf{A}}$ is the derivation of $\mathbf{A}$ with respect to θ. For the case of 2 orthogonal signals, the minimal CRB is obtained by $|\beta| = 0$, which represents the case of orthogonal transmitted signals. The design of orthogonal signals is discussed in [1].

The DOA estimation performance with incoherent transmitted signals is superior comparing to the case of correlated transmitted signals, because transmitting incoherent signals enables to extend the virtual aperture of the array, which narrows the beam shape. In the case of $\theta = 0°$, there is no loss in the gain of the array beam, henceforth the performance will be the same for any value of the correlation factor.

5. SIMULATION RESULTS

In this section, we demonstrate via simulations the detection and localization performance for the case of spatially coded signals. An array of two elements with half a wavelength spacing was chosen. Fig. 3 shows the square root of the CRB for different angles: $0^o, 20^o, 30^o, 40^o$ as a function of correlation coefficient $|\beta|$. It is evident that the CRB is minimized at $|\beta| = 0$. For $\theta = 0^o$, the CRB is constant for all correlation factors. It can also be seen that the CRB is increased monotonically with $|\beta|$. Fig. 4 presents the root-mean-square-error (RMSE) for target localization and CRB for different values of $|\beta|$: 0, 0.5, 1 as a function of SNR. It can be seen that the RMSE increases with $|\beta|$. This due to the beam shape loss in transmission when

$|\beta| \neq 0$. In the case of $|\beta| = 0$, the algorithm steers the transmitted beam in teh receive mode and therefore beam shape loss is avoided. In Fig. 5 Receiver Operation Characteristic (ROC) curves are presented using theoretical and simulation results versus different values of $|\beta|$: 0, 0.5, 1, for $SNR = 10\ dB$.

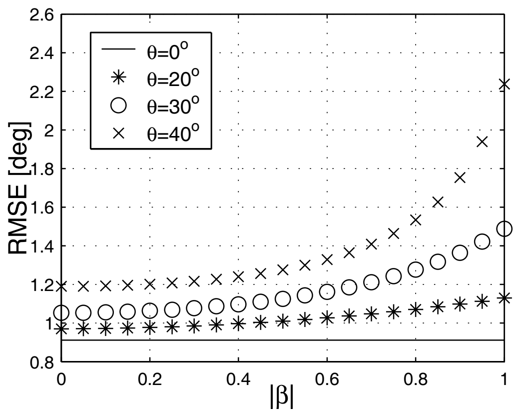

Figure 3: CRB for different correlation coefficients

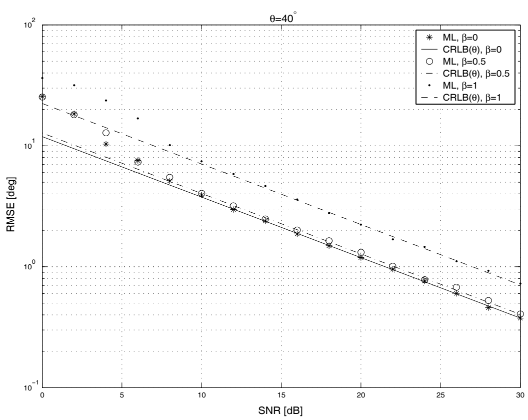

Figure 4: ML and CRB as a function of SNR for different correlation coefficients

6. DISCUSSION AND CONCLUSIONS

We presented a new approach for signal transmission in radar and sonar systems, which enables to avoid beam-shape loss by transmitting different, incoherent signals. The proposed method allows to extend the virtual aperture of the array, enabling to generate narrower beam-width and to improve the angular resolution. On the other hand, because of incoherent transmitted signals, the gain of the main beam is lower than in the conventional beamforming method. The target localization performance of the proposed configuration is superior compared to the conventional one when the signal is not in the center of the beam.

Another important property of the new model is the time required to scan a given region of interest (ROI). In conventional beamforming, the scan time of the whole ROI is the time required to cover the ROI with directed, narrow and overlapped beams. In the new approach, the wide

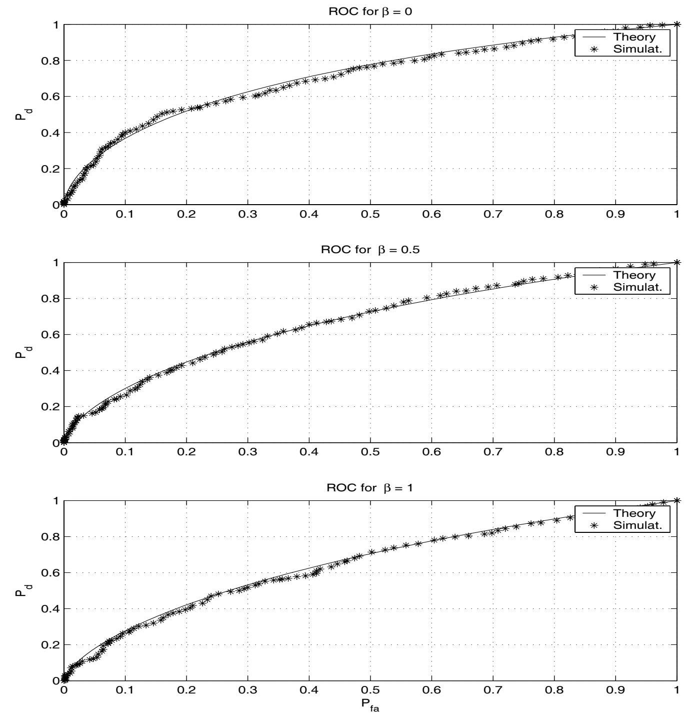

Figure 5: ROC for different correlation coefficients

or omni-directional beams, enable to enlarge the time-on-target period during the scan time of conventional beamforming. The target detection performance is better than the conventional model because in the new configuration,the beam overlaps and beam-shape-loss can be avoided.

In practice, multiple matched filtering for each sensor, as presented in Fig. 2, may be complicated and be expensive in systems with large number of sensors. Therefore, in such systems the new configuration may be implemented by division of array of sensors into sub-arrays, transmitting different signals.

7. REFERENCES

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