ROBUST SPACE–TIME BEAMFORMING IN GNSS BY MEANS OF SECOND–ORDER CONE PROGRAMMING

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ABSTRACT

This paper deals with the reception of several Global Navigation Satellite Systems (GNSS) signals using antenna arrays in the presence of interferences, multipath propagation and array manifold uncertainties. The proposed method is a new space–time approach where all the Line–Of–Sight Signals (LOSS) are pointed by the array, and the temporal reference of the signal is exploited. The work also presents an extension of this beamforming that takes into account the incertitudes in the steering matrix due to pointing errors or array miscalibration. The robustness of this new method is achieved using second–order cone programming, allowing its implementation with highly efficient algorithms. Simulations show the performance of these beamformers in scenarios where coherent multipath and pointing errors are present.

1. INTRODUCTION

Beamforming with antenna arrays consists of several antennas which outputs are controlled in phase and gain, *i.e.*, multiplied by complex weights, in order to achieve a gain pattern that can be manipulated electronically. Then, all the weighted signals are combined to obtain a single output that feeds a conventional single–antenna receiver. Traditional approaches devoted to GNSS are based in the weakness of the received signal; it cannot be detected or measured without a correlation process because is under the noise floor (in GPS, the $L1$ signal has a minimum of -160 dBw at the Earth's surface). Therefore, processing algorithms assume that any measurable energy above the noise must be a jamming signal, so the weights are calculated to null the direction of arrival of the interference. These techniques are effective against a limited number of narrowband interferences, but hardly could cope with broadband interferences or multipath. More jammers can be mitigated by space-time adaptive processing techniques exploiting spatial and temporal information. Conventional space-time (likewise space-frequency) processing is also based upon the assumption that any measured power must be a jamming signal. Thus, a GNSS signal may be attenuated because its Direction Of Arrival (DOA) is not taken into account.

But the DOA is not a completely unknown information in GNSS. Satellites are continuously broadcasting a low rate navigation message which allows to estimate their position. Hence, a rough estimation of the array position and attitude may be enough to determine the angle of incidence of the desired signals. The sources are about $20,000$ Km far from the receiver, so the accuracy in the preliminary receiver position estimation is not a critical issue. Errors in the attitude determination appears to have a greater impact in DOA estimation. These errors are ascribable to the Inertial Measurement Unit (IMU) used; a low–cost Micro Electro Mechanical System (MEMS) has an accuracy of about ± 5 degrees and a drift rate of 3 degrees/hour. Even with a high–accuracy IMU, pointing errors due to an array miscalibration could not be dismissable at all, provided that array systems are known to be quite sensitive to mismatches between the presumed and the actual DOA.

2. PROBLEM FORMULATION

An N–element antenna array receives M scaled, time–delayed and Doppler–shifted signals with known structure. The receiving complex baseband signal can be modeled as

$$x(t) = \sum_{i=1}^{M} a_i s_i(t-\tau_i) \exp\{j2\pi f_i t\} + w(t) \qquad (1)$$

where a_i is the complex amplitude of each signal, τ_i is the delay, f_i is the Doppler shift, and $w(t)$ represents additive white Gaussian noise and all other disturbing terms.

Each antenna receives a different replica of this set of signals, with a different phase depending on the array geometry and the Directions Of Arrival (DOA). This can be expressed by a vector signal model, where each row corresponds to one antenna:

$$\mathbf{x}(t) = \mathbf{G}(\boldsymbol{\theta}, \boldsymbol{\phi})\mathbf{A}\mathbf{d}(t, \boldsymbol{\tau}, \mathbf{f}) + \mathbf{n}(t) \qquad (2)$$

where[1]

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[1] The transpose, conjugate and conjugate transpose operations are des-

- $\mathbf{x}(t) \in \mathbb{C}^{N\times 1}$ is the observed signal vector,
- $\mathbf{G}(\boldsymbol{\theta},\boldsymbol{\phi}) \in \mathbb{C}^{N\times M}$, called steering matrix, is related to the array geometry and the DOA (elevation $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 & \dots & \theta_M \end{bmatrix}^T$ and azimuth $\boldsymbol{\phi} = \begin{bmatrix} \phi_1 & \dots & \phi_M \end{bmatrix}^T$ vectors). It may also depend on additional parameters, such as scattering, polarization or calibration coefficients.
- $\mathbf{A} = diag(\mathbf{a}) \in \mathbb{C}^{M\times M}$ is a diagonal matrix with the elements of the amplitude vector $\mathbf{a} = [a_1 \cdots a_M]^T$ along its diagonal,
- $\mathbf{d}(t,\boldsymbol{\tau},\mathbf{f}) = \begin{bmatrix} s_1(t-\tau_1)\exp\{j2\pi f_1 t\} \\ \vdots \\ s_M(t-\tau_M)\exp\{j2\pi f_M t\} \end{bmatrix}$,
  $\mathbf{d} \in \mathbb{C}^{M\times 1}$ the delayed Doppler–shifted narrowband signals envelopes, where $\boldsymbol{\tau} = \begin{bmatrix} \tau_1 & \dots & \tau_M \end{bmatrix}^T$ and $\mathbf{f} = \begin{bmatrix} f_1 & \dots & f_M \end{bmatrix}^T$, and
- $\mathbf{n}(t) \in \mathbb{C}^{N\times 1}$ represents additive noise and all other disturbing terms, such as interferences or multipath of each signal. In GNSS, only the LOSS are signals of interest.

This model is built upon the *narrowband array assumption*, consisting of taking the time required for the signal to propagate along the array as much smaller than its inverse bandwidth. Thus, a phase shift can be used to describe the propagation from one antenna to another. Taking the $L1$ signal specification of the GPS Standard Positioning System [1] (right hand circularly polarized BPSK signal modulated by a 1.023 MHz Gold code, filtered at 20 MHz and with a 1575.42 MHz carrier), we have an inverse passband bandwidth of approximately $5\cdot 10^{-8}$ s, although in commercial front–ends the signal is often limited to 2 MHz ($5\cdot 10^{-7}$ s). A signal coming from the end–fire propagating along an 8–element linear antenna array with a half wavelength between antennas takes $3\cdot 10^{-9}$ s, and thus the assumption holds.

In the same way, we have assumed that the Doppler effect can be modeled by a frequency shift, which is commonly referred as the *narrowband signal assumption*. Due to the satellite constellation characteristics, the maximum expected Doppler frequency shift in a GPS receiver is about ± 5 KHz, much less than the signal bandwidth, and again the proposed assumption seems reasonable. Similar values are expected in the forthcoming Galileo system, the next step in GNSS.

Suppose that K snapshots of the impinging signal are taken at a suitable rate. Then the sampled data can be expressed as

$$\mathbf{X} = \mathbf{GAD} + \mathbf{N} \tag{3}$$

ignated by $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ respectively. $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denote real and imaginary parts

using the following definitions:

- $\mathbf{X} = \begin{bmatrix} \mathbf{x}(t_0) & \cdots & \mathbf{x}(t_{K-1}) \end{bmatrix} \in \mathbb{C}^{N\times K}$
- $\mathbf{D} = \begin{bmatrix} \mathbf{d}(t_0) & \dots & \mathbf{d}(t_{K-1}) \end{bmatrix} \in \mathbb{C}^{M\times K}$
- $\mathbf{N} = \begin{bmatrix} \mathbf{n}(t_0) & \cdots & \mathbf{n}(t_{K-1}) \end{bmatrix} \in \mathbb{C}^{N\times K}$

3. HYBRID SPACE–TIME REFERENCE BEAMFORMING

This section presents a new type of (multiple) beamforming that exploits *a priori* DOA information, and where the pointing errors are neglected. Firstly, we define the following notation based on the signal model (3):

$$\begin{array}{ll} \hat{\mathbf{R}}_{XX} = \frac{1}{K}\mathbf{X}\mathbf{X}^H & \hat{\mathbf{R}}_{XD} = \frac{1}{K}\mathbf{X}\mathbf{D}^H \\ \hat{\mathbf{R}}_{DX} = \hat{\mathbf{R}}_{XD}^H & \hat{\mathbf{R}}_{DD} = \frac{1}{K}\mathbf{D}\mathbf{D}^H \end{array} \tag{4}$$

and

$$\hat{\mathbf{W}} = \hat{\mathbf{R}}_{XX} - \hat{\mathbf{R}}_{XD}\hat{\mathbf{R}}_{DD}^{-1}\hat{\mathbf{R}}_{XD}^H \tag{5}$$

The mean square error (MSE) between the output of a beamformer with weights $\mathbf{w}$ and a temporal reference signal $\mathbf{a}^T\mathbf{D}$ is

$$J_1(\mathbf{w}) = \frac{1}{K}\left\|\mathbf{w}^H\mathbf{X} - \mathbf{a}^T\mathbf{D}\right\|^2 \tag{6}$$

In this case, the temporal reference is not completely known but parametrized by the amplitudes $\mathbf{a}$, the Doppler shifts $\mathbf{f}$ and the time delays $\boldsymbol{\tau}$. In order to take advantage of the knowledge of the steering matrix, a spatial constraint is imposed to force the beamformer to always point the signals. This combination of temporal and spatial information can be called multiple hybrid beamformer, and this new criterion can be stated as follows:

$$\min_{\mathbf{w}} J_1(\mathbf{w}) \tag{7}$$

$$\text{subject to} \quad \mathbf{w}^H\mathbf{G} = \mathbf{1}_{1\times M} \tag{8}$$

The amplitudes vector that minimizes J_1 for fixed $\mathbf{w}$, $\mathbf{f}$ and $\boldsymbol{\tau}$ is

$$\hat{\mathbf{a}}_{MHB} = \left(\mathbf{D}^*\mathbf{D}^T\right)^{-1}\mathbf{D}^*\mathbf{X}^T\mathbf{w}^* \tag{9}$$

Replacing (9) in (6) we obtain a new cost function that has to be minimized

$$J_2(\mathbf{w}) = \mathbf{w}^H\mathbf{W}\mathbf{w} \tag{10}$$

being $\mathbf{W}$ defined as in equation (5). This is a well-known M linear–constrained (8) quadratic–form (10) optimization problem. Applying Lagrange's multipliers technique, the optimum weight vector is

$$L = J_1 + \mathbf{w}^H\mathbf{G}\boldsymbol{\lambda} \tag{11}$$

and making its derivative respect to $\mathbf{w}^*$ equal to the zero vector:

$$\nabla_{\mathbf{w}^*}\left(L\left(\mathbf{w}_{MHB}\right)\right) = \hat{\mathbf{R}}_{\mathbf{xx}}\mathbf{w}_{MHB} - \hat{\mathbf{R}}_{\mathbf{xD}}\mathbf{a}^* + \mathbf{G}\boldsymbol{\lambda} = \mathbf{0} \tag{12}$$

we obtain the weight vector which minimizes (7):

$$\hat{\mathbf{w}}_{MHB} = \hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\left(\hat{\mathbf{R}}_{\mathbf{xD}}\mathbf{a}^* - \mathbf{G}\boldsymbol{\lambda}\right) \tag{13}$$

Applying to (13) the spatial constraints defined in (8), Lagrange multipliers take the form

$$\boldsymbol{\lambda} = -\left(\mathbf{G}^H\hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\mathbf{G}\right)^{-1}\left(\mathbf{1} - \mathbf{G}^H\hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\hat{\mathbf{R}}_{\mathbf{xD}}\mathbf{a}^*\right) \tag{14}$$

Finally, inserting (14) in (13), a very interesting expression for the weight vector is obtained:

$$\begin{aligned}\hat{\mathbf{w}}_{MHB} &= \hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\hat{\mathbf{R}}_{\mathbf{xD}}\mathbf{a}^* + \\ &+ \hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\mathbf{G}\left(\mathbf{G}^H\hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\mathbf{G}\right)^{-1}\left(\mathbf{1} - \mathbf{G}^H\hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\hat{\mathbf{R}}_{\mathbf{xD}}\mathbf{a}^*\right)\end{aligned} \tag{15}$$

This result is a multiple beamforming which is a linear combination of two previously known results. On one hand,

$$\hat{\mathbf{w}}_{TE} = \hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\hat{\mathbf{R}}_{\mathbf{xD}}\mathbf{a}^* \tag{16}$$

is the multiple beamforming under the MSE criterion taking into account only the temporal reference. On the other hand,

$$\hat{\mathbf{w}}_{MVB} = \hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\mathbf{G}\left(\mathbf{G}^H\hat{\mathbf{R}}_{\mathbf{xx}}^{-1}\mathbf{G}\right)^{-1}\mathbf{1} \tag{17}$$

is the minimum variance beamforming [2] considering only the spatial information. These solutions have a different behavior against multipath and interferences: while $\mathbf{w}_{TE}$ tries to combine constructively the LOSS with the reflections in order to increase the SNIR, $\mathbf{w}_{MVB}$ combines destructively such signals to minimize the output signal power [3]. The multiple hybrid beamforming combines these two behaviors to mitigate multipath and interferences.

4. ROBUST HYBRID SPACE–TIME REFERENCE BEAMFORMING

The objective of this section is to obtain a beamforming that minimizes the estimation error exploiting the prior DOA information but taking into account the mentioned uncertainties in the steering matrix. Let $\mathbf{G}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}) = \mathbf{G}(\boldsymbol{\theta}, \boldsymbol{\phi}) + \mathbf{E}$, where $\boldsymbol{\theta}, \boldsymbol{\phi}$ are the *true* DOAs and the error matrix $\mathbf{E}$ contains the distortions in the DOA estimations. In practical applications, we assume that the error matrix $\mathbf{E}$ can be bounded in some sense; we propose the Frobenius norm

$$||\mathbf{E}||_F \leq \epsilon, \quad \epsilon > 0 \tag{18}$$

The idea is to impose to the plane waves arriving from directions contained in the region defined by $\mathbf{E}$ an amplification gain greater than or equal to unity, following the concept explained in [4]. This leads to a reformulation of the problem:

$$\min_{\mathbf{w}} \mathbf{w}^H\mathbf{W}\mathbf{w} \tag{19}$$

$$\text{subject to} \quad \mathbf{w}^H\mathbf{S} \geq \mathbf{1}_{1\times M} \quad \forall \mathbf{S} \in \mathcal{S}(\epsilon) \tag{20}$$

where

$$\mathcal{S}(\epsilon) = \{\mathbf{S}|\mathbf{S} = \mathbf{G}(\boldsymbol{\theta}, \boldsymbol{\phi}) + \mathbf{E}, ||\mathbf{E}||_F \leq \epsilon\} \tag{21}$$

Using (21) we can express the constraint (20) as M different constraints in the form

$$\min_{e\in\{\mathbf{e}|\ ||\mathbf{e}||\leq\epsilon_i\}} ||\mathbf{w}^H\mathbf{g}_i + \mathbf{w}^H\mathbf{e}_i|| \leq 1, \quad i = 1...M \tag{22}$$

where $\mathbf{g}_i$ and $\mathbf{e}_i$ are the column vectors of $\mathbf{G}$ and $\mathbf{E}$ and ϵ_i is the norm of $\mathbf{e}_i$. Note that $\epsilon = \sqrt{\sum_{i=1}^{M}\epsilon_i^2}$.

Cauchy–Schwarz inequality says that

$$||\mathbf{w}^H\mathbf{g}_i + \mathbf{w}^H\mathbf{e}_i|| \geq ||\mathbf{w}^H\mathbf{g}_i|| - ||\mathbf{w}^H\mathbf{e}_i|| \geq ||\mathbf{w}^H\mathbf{g}_i|| - \epsilon_i||\mathbf{w}^H|| \tag{23}$$

and this expression allows an equivalent formulation of the problem:

$$\min_{\mathbf{w}} \mathbf{w}^H\mathbf{W}\mathbf{w} \tag{24}$$

$$\text{subject to} \quad \epsilon_i||\mathbf{w}^H|| \leq \mathbf{w}^H\mathbf{g}_i - 1 \quad i = 1...M \tag{25}$$

We have obtained a set of M second–order cone constraints [5]. For the ease of programming, complex quantities can be transformed into real values with the following definitions:

$$\breve{\mathbf{w}} = \begin{bmatrix} \Re\{\mathbf{w}\} \\ \Im\{\mathbf{w}\} \end{bmatrix}, \breve{\mathbf{g}}_i = \begin{bmatrix} \Re\{\mathbf{g}_i\} \\ \Im\{\mathbf{g}_i\} \end{bmatrix}, \quad \breve{\mathbf{W}} = \begin{bmatrix} \Re\{\mathbf{W}\} & -\Im\{\mathbf{W}\} \\ \Im\{\mathbf{W}\} & \Re\{\mathbf{W}\} \end{bmatrix} \tag{26}$$

which permit to express $\mathbf{w}^H\mathbf{W}\mathbf{w}$ as $\breve{\mathbf{w}}^T\breve{\mathbf{W}}\breve{\mathbf{w}}$.

In order to cast the problem into a second–order cone program and to benefit of its simple implementation, the quadratic objective function $\breve{\mathbf{w}}^T\breve{\mathbf{W}}\breve{\mathbf{w}}$ can be easily linearized. Assuming that $\breve{\mathbf{W}}$ is positive definite, $\breve{\mathbf{W}} = \mathbf{U}^H\mathbf{U}$ is its Cholesky factorization, and thus equation (10) can be expressed as $\breve{\mathbf{w}}^T\breve{\mathbf{W}}\breve{\mathbf{w}} = ||\mathbf{U}\breve{\mathbf{w}}||^2$, and the minimization of $||\mathbf{U}\breve{\mathbf{w}}||^2$ is equivalent to the minimization of $||\mathbf{U}\breve{\mathbf{w}}||$. Hence, we can convert (24) into a linear objective function simply defining a nonnegative scalar ξ and a new (convex) constraint $||\mathbf{U}\breve{\mathbf{w}}|| \leq \xi$:

$$\min_{\mathbf{w}} \xi \tag{27}$$

$$\text{subject to} \quad ||\mathbf{U}\breve{\mathbf{w}}|| \leq \xi \tag{28}$$

The problem at hand can be rewritten as:

$$\min_{\check{\mathbf{w}}} \xi \tag{29}$$

$$\text{subject to} \quad \|\mathbf{U}\check{\mathbf{w}}\| \le \xi, \tag{30}$$

$$\begin{aligned} \epsilon_1\|\check{\mathbf{w}}\| &\le \check{\mathbf{w}}^T\check{\mathbf{g}}_1 - 1, \\ &\vdots \\ \epsilon_M\|\check{\mathbf{w}}\| &\le \check{\mathbf{w}}^T\check{\mathbf{g}}_M - 1 \end{aligned}$$

which is a second–order cone program. The objective (29) is a convex function and the constraints (30) define a convex set. Thus, this is a convex programming problem which can be solved efficiently in polynomial time via interior point algorithms [5] .

5. SIMULATION RESULTS

An 8–element uniform linear array with half–wavelength antenna spacing is considered. The simulation scenario consists of a GPS–like BPSK signal centered at 1575.42 MHz (1 KHz Doppler–shifted), with $CN0 = 30$ dB, a bit rate of 1023 Kbps and taken at 4 samples per bit. It impinges the array at 45° from the broadside. An echo coming from $-45°$, $CN0 = 20$ dB and with a time delay of a half a bit (coherent multipath) with respect to the LOSS and -2 KHz Doppler shift is also present. In figure 1 the performance of the two proposed beamformings in case of perfectly known steering matrix is shown. The same scenario can be viewed in figure 2, but considering a mismatch of 5° in the desired DOA. Both methods are compared with the optimal beamforming, *i.e.*, with exact knowledge of matrix $\mathbf{N}$ (unavailable in practical applications). Each simulated point corresponds to the average of 100 independent realizations. In both examples, the proposed robust hybrid beamformer effectively mitigates coherent multipath and attains a better performance than the hybrid beamforming, specially in presence of pointing errors.

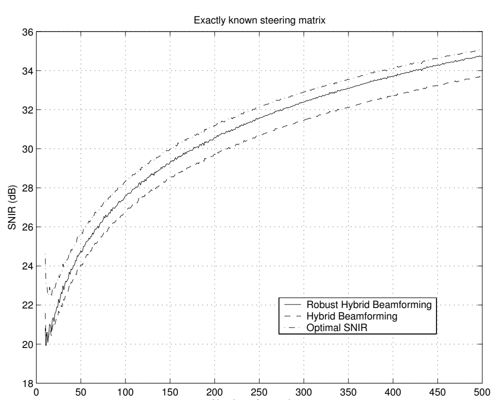

Fig. 1. Output SNIR for known steering matrix

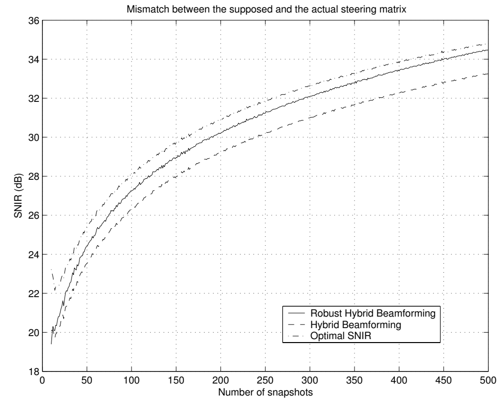

Fig. 2. Output SNIR with a steering mismatch of 5°

6. CONCLUSIONS

This paper presents a space–time approach to multiple beamforming which exploits the availability of *a priori* Directions Of Arrival information in GNSS mitigating coherent multipath and interferences. A robust version of this algorithm, where modeling array uncertainties has been taken into account, has been also derived. The resulting problem can be cast in a second–order cone program, a convex optimization problem which can be solved efficiently by interior point algorithms in polynomial time. Numerical results confirm the effectiveness of the proposed beamformers.

7. REFERENCES

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