NOVEL CLOSED-FORM ML POSITION ESTIMATOR FOR HYPERBOLIC LOCATION

Andreu Urruela and Jaume Riba

Signal Processing and Communications group. Technical University of Catalonia (UPC)
Campus Nord, Ed. D5, Jordi Girona 1-3, 08034, Barcelona (SPAIN).
Email: {andreu,jriba}@gps.tsc.upc.es

ABSTRACT

[1] Geolocation of mobile terminals has become in the last decades an important issue in mobile networks. In the literature, there have been presented several closed-form position estimators based on Time-difference-of-arrival (TDOA) measurements. Only Fang's estimator can be considered optimum in the Maximum Likelihood (ML) sense. Unfortunately, it can only be applied to the particular case of two TDOA measurements for the two dimensional (2D) location case. This paper presents an extension of this closed-form estimator to be applied to an arbitrary number of TDOA measurements by means of a transformation in the Maximum Likelihood function. This allows to split the ML function minimization in several partial ML minimizations which only consider a subset of the available measurements where the original Fang's estimator can be applied. Numerical simulation show that the proposed algorithm, that can be considered asymptotically the ML-estimator, attains the theoretical limits for all range of reasonable SNR values and has a low implementation complexity.

1. INTRODUCTION

A problem of growing importance in mobile communication networks is finding the position of mobile terminals. This will be mandatory for public-access cell-networks and very useful for future position-based services and position-based network managers. One of the most common approaches is getting the position estimate by measuring several time of arrival (TOA) or TDOA measurements among a set of neighbor base stations (BSs). The major problem in this kind of approaches is the non-linear relationship between the measurements (TDOAs or TOAs) and the position (Cartesian coordinates). This phenomenon produces a non-convex ML function difficult to maximize.

In the recent literature, several approaches have been presented to solve this problem. The most common approach is the implementation of iterative algorithms [1] where the position estimate is improved at each step by finding the local minimum Least Square (LS) solution (using the linearized ML-function). The major problems of this approach are the high complexity associated to the LS procedure and the difficulty of finding an initial estimate for the position.

Another approach presented in previous publications consists in finding closed-form expressions for the position estimate. This has the great advantage that the position is delivered in one step and an initial position estimate is not required. Contributions in this direction can be found in [2], [3] and [4] where several approximations are performed for linearizing the problem. Although good performance is obtained with these closed-form estimators, especially with a high number of TDOA measurements, they can not be considered optimum in the ML sense.

The major contribution in this direction was performed by Fang [5] who obtained a closed-form position estimator in the form of a simple quadratic equation. This is found by solving the equations related with the hyperbolas associated to the TDOA measurements. The main advantage of this algorithm is its optimality due to the fact that it is equivalent, as it will be shown in this paper, to the ML algorithm. The major drawback of this algorithm is that it only works when the minimum number of TDOAs are presented (i.e. 2 TDOA measurements for a 2D location). Unfortunately, as far as we know, there has not been presented in the literature an extension of his work for the generic case of an arbitrary number of TDOA measurements.

The contribution of this paper is generalizing the explicit closed-form estimator presented by Fang in [5] for the generic case of N independent TDOA measurements. This will be able thanks to a transformation of the ML non-convex function that allows to estimate the position as a linear combination of partial estimates. These are obtained applying the original Fang's estimator to a subset of the original TDOA measurements. This new estimator is shown to be asymptotically (in time) the ML estimator which attains asymptotically the Cramer-Rao-Lower-Bound (CRB). Simulation results will show that the new estimator attains the theoretical limits, presents good performance in all reasonable range of SNR and has a low implementation complexity.

2. PROBLEM STATEMENT

Assuming that $\mathbf{t} = [t_1, \cdots, t_N]^T$ is the vector containing the N independent TDOA measurements from a set of $L = N+1$ BSs and $\mathbf{R_t} = diag(\sigma_1^2, \cdots, \sigma_N^2)$ is the covariance matrix of $\mathbf{t}$, the ML position estimator in the case of

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gaussian noise can be expressed as follows:

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x}} \phi(\mathbf{x}, \mathbf{t}) \tag{1}$$

where $\phi(\mathbf{x}, \mathbf{t})$ is the negative log-likelihood function defined as:

$$\phi(\mathbf{x}, \mathbf{t}) = (\mathbf{t} - \mathbf{f}(\mathbf{x}))^T \mathbf{R_t}^{-1} (\mathbf{t} - \mathbf{f}(\mathbf{x})) \tag{2}$$

where $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \cdots, f_N(\mathbf{x})]^T$ is a vector containing the well known non-linear hyperbolic functions (for the 2D case):

$$f_n(\mathbf{x}) = \left\| \mathbf{x} - \mathbf{x}_{p_{n,1}} \right\| - \left\| \mathbf{x} - \mathbf{x}_{p_{n,2}} \right\| \tag{3}$$

where $p_{n,1}$ and $p_{n,2}$ are the indexes of the associated BSs of the n-th TDOA measurement. Normally these indexes are $p_{n,1} = 1$ and $p_{n,2} = n + 1$, which correspond to the case when all TDOA measurements are performed with respect to a common BS. In any case, $1 \leq p_{n,1}$,$p_{n,2} \leq L$.

Note in (2) that if we can find a position estimate $(\widehat{\mathbf{x}}_F)$ that satisfies $\mathbf{f}(\widehat{\mathbf{x}}_F) = \mathbf{t}$ independently of the value of $\mathbf{R_t}$, this will be always the ML estimation because it will be always the minimizer of $\phi(\mathbf{x}, \mathbf{t})$. This equation was solved actually by Fang [5] and this is why the Fang's closed-form position estimator can be considered optimum in the ML sense. Unfortunately, this estimate can only be found when the number of TDOA measurements is equal to the number of Cartesian coordinates of the position to be estimated ($N = 2$ for the 2D case). With additional TDOA measurements, there will not be solution for the equation $\mathbf{f}(\widehat{\mathbf{x}}_F) = \mathbf{t}$.

3. EXTENSION OF THE ML APPROACH

In this section, it will be presented a generalized version of the ML non-convex function $\phi(\mathbf{x}, \mathbf{t})$ defined in (2), needed to divide the ML position estimate $\widehat{\mathbf{x}}$ into a linear combination of partial ML estimators. These partial ML estimators will be solved with the Fang's closed-form algorithm that is actually the ML estimator for a specific number of TDOA measurements as it has been shown in the previous section.

First of all, it is not difficult to show that $\phi(\mathbf{x}, \mathbf{t})$ can be expressed as:

$$\phi(\mathbf{x}, \mathbf{t}) = (\mathbf{t}_e - \mathbf{f}_e(\mathbf{x}))^T \mathbf{R}_{\mathbf{t}_e}^{-1} (\mathbf{t}_e - \mathbf{f}_e(\mathbf{x})) \tag{4}$$

where

$$\begin{aligned} \mathbf{t}_e &= \left[t_1 \cdot \mathbf{1}_{Q_1}^T, \cdots, t_N \cdot \mathbf{1}_{Q_N}^T\right]^T & (5)\\ \mathbf{f}_e(\mathbf{x}) &= \left[f_1(\mathbf{x}) \cdot \mathbf{1}_{Q_1}^T, \cdots, f_N(\mathbf{x}) \cdot \mathbf{1}_{Q_N}^T\right]^T & (6)\\ \mathbf{R}_{\mathbf{t}_e} &= \mathrm{diag}\left(\sigma_1^2 \cdot \mathrm{Q}_1 \cdot \mathbf{I}_{\mathrm{Q}_1}, \cdots, \sigma_\mathrm{N}^2 \cdot \mathrm{Q}_\mathrm{N} \cdot \mathbf{I}_{\mathrm{Q}_\mathrm{N}}\right) & (7) \end{aligned}$$

where $\mathbf{1}_p$ is an all-ones p-length column vector, $\mathbf{I}_p$ is the identity matrix of rank p, and $Q_n > 0$ is the positive integer indicating the number of times that the contribution of the n-th TDOA measurement (t_n) is repeated in $\mathbf{t}_e$, $\mathbf{f}_e(\mathbf{x})$ and $\mathbf{R}_{\mathbf{t}_e}$. Note here that the dimension of the vectors $\mathbf{t}_e$, $\mathbf{f}_e(\mathbf{x})$ and the square matrix $\mathbf{R}_{\mathbf{t}_e}$ are $M = \sum_{n=1}^{N} Q_n$ and MxM respectively.

This first transformation allows us to express the contribution of a specific TDOA measurement (the n-th for instance) as the independent contributions of Q_n virtual TDOA measurements with the same value of the original one and with a variance Q_n times higher.

The second steep is to realize that a permutation order does not affect the ML-function. So we have:

$$\phi(\mathbf{x}, \mathbf{t}) = (\mathbf{t}_{ep} - \mathbf{f}_{ep}(\mathbf{x}))^T \mathbf{R}_{\mathbf{t}_{ep}}^{-1} (\mathbf{t}_{ep} - \mathbf{f}_{ep}(\mathbf{x})) \tag{8}$$

where

$$\mathbf{t}_{ep} = \mathbf{P}_m \mathbf{t}_e \quad \mathbf{f}_{ep}(\mathbf{x}) = \mathbf{P}_m \mathbf{f}_e(\mathbf{x}) \quad \mathbf{R}_{\mathbf{t}_{ep}} = \mathbf{P}_m \mathbf{R}_{\mathbf{t}_e} \mathbf{P}_m^T \tag{9}$$

$\mathbf{P}_m$ being any square permutation matrix of dimension M.

Finally, it can be demonstrated that if the M elements of the vectors $\mathbf{t}_{ep}$ and $\mathbf{f}_{ep}(\mathbf{x})$ are partitioned into a set of K subvectors of generic lengths as follows:

$$\mathbf{t}_{ep} = \left[\mathbf{t}_{ep_1}{}^T, \cdots, \mathbf{t}_{ep_K}{}^T\right]^T \tag{10}$$

$$\mathbf{f}_{ep}(\mathbf{x}) = \left[\mathbf{f}_{ep_1}^T(\mathbf{x}), \cdots, \mathbf{f}_{ep_K}^T(\mathbf{x})\right]^T \tag{11}$$

and if the square diagonal matrix $\mathbf{R}_{\mathbf{t}_{ep}}$ is accordingly partitioned in K square diagonal submatrices as:

$$\mathbf{R}_{\mathbf{t}_{ep}} = \mathrm{diag}\left(\mathbf{R}_{\mathbf{t}_{\mathrm{ep}_1}}, \cdots, \mathbf{R}_{\mathbf{t}_{\mathrm{ep}_\mathrm{K}}}\right) \tag{12}$$

the ML position estimate $\widehat{\mathbf{x}}$ shown in (1), that is the minimizer of $\phi(\mathbf{x}, \mathbf{t})$, can be approximated as follows:

$$\widehat{\mathbf{x}} \approx \left[\sum_{k=1}^{K} \mathbf{C}_k^{-1}(\widehat{\mathbf{x}}_k)\right]^{-1} \sum_{k=1}^{K} \mathbf{C}_k^{-1}(\widehat{\mathbf{x}}_k) \cdot \widehat{\mathbf{x}}_k \tag{13}$$

where $\widehat{\mathbf{x}}_k$ is the ML position estimate using only the TDOA measurements included in the vector $\mathbf{t}_{ep_k}$, with the associated covariance matrix $\mathbf{R}_{\mathbf{t}_{ep_k}}$. A similar expression to (13) can be found in the case of a non block-diagonal matrix $\mathbf{R}_{\mathbf{t}_{ep}}$ so the initial diagonal assumption for $\mathbf{R_t}$ is not a restriction in this approach.

To demonstrate this last step (to obtain (13)) and to find the expression of $\mathbf{C}_k(\mathbf{x})$, let us apply the definitions of the partition, this is (10), (11) and (12) to the ML-function shown in (8) to obtain:

$$\phi(\mathbf{x}, \mathbf{t}) = \sum_{k=1}^{K} \left(\mathbf{t}_{ep_k} - \mathbf{f}_{ep_k}(\mathbf{x})\right)^T \mathbf{R}_{\mathbf{t}_{ep_k}}^{-1} \left(\mathbf{t}_{ep_k} - \mathbf{f}_{ep_k}(\mathbf{x})\right) \tag{14}$$

Now, assuming the following linear approximation

$$\mathbf{f}_{ep_k}(\mathbf{x}) \approx \mathbf{f}_{ep_k}(\mathbf{x}_{0k}) + \mathbf{F}_{ep_k}(\mathbf{x}_{0k})[\mathbf{x} - \mathbf{x}_{0k}] \tag{15}$$

where we define $\mathbf{F}_{ep_k}(\mathbf{x}) = \nabla_\mathbf{x} \mathbf{f}_{ep_k}(\mathbf{x})$ as the gradient of $\mathbf{f}_{ep_k}(\mathbf{x})$ w.r.t. $\mathbf{x}$, we can obtain the ML position estimator solving the equation $\nabla_\mathbf{x} \phi(\mathbf{x}, \mathbf{t}) = \mathbf{0}$ as follows:

$$\widehat{\mathbf{x}} \approx \left[\sum_{k=1}^{K} \mathbf{C}_k^{-1}(\mathbf{x}_{0k})\right]^{-1} \sum_{k=1}^{K} \mathbf{C}_k^{-1}(\mathbf{x}_{0k}) \cdot \widehat{\mathbf{x}}_k \tag{16}$$

where

$$\mathbf{C}_k(\mathbf{x}) = \left(\mathbf{F}_{ep_k}(\mathbf{x})^T \mathbf{R}_{\mathbf{t}_{ep_k}}^{-1} \mathbf{F}_{ep_k}(\mathbf{x})\right)^{-1} \tag{17}$$

and

$$\widehat{\mathbf{x}}_k = \mathbf{x}_{0k}+\mathbf{C}_k(\mathbf{x}_{0k})\cdot\mathbf{F}_{ep_k}(\mathbf{x}_{0k})\cdot\mathbf{R}^{-1}_{\mathbf{t}_{ep_k}}\left(\mathbf{t}_{ep_k}-\mathbf{f}_{ep_k}(\mathbf{x}_{0k})\right) \tag{18}$$

Note here, that $\widehat{\mathbf{x}}_k$ is the approximated ML position estimator only taking into account the TDOA measurements of the k-th subset $\mathbf{t}_{ep_k}$ and their associated variances $\mathbf{R}_{\mathbf{t}_{ep_k}}$ under the linear approximation in (15). This can be simply substituted by the natural ML estimator defined as follows:

$$\widehat{\mathbf{x}}_k = \arg\min_{\mathbf{x}} \phi\left(\mathbf{x}, \mathbf{t}_{ep_k}\right) \tag{19}$$

$$\phi\left(\mathbf{x}, \mathbf{t}_{ep_k}\right) = \left(\mathbf{t}_{ep_k} - \mathbf{f}_{ep_k}(\mathbf{x})\right)^T \mathbf{R}^{-1}_{\mathbf{t}_{ep_k}} \left(\mathbf{t}_{ep_k} - \mathbf{f}_{ep_k}(\mathbf{x})\right) \tag{20}$$

Finally, it can be noted that matrices $\mathbf{C}_k(\mathbf{x})$ in (16) depend on a position estimate $\mathbf{x}_{0k}$ (defined as a position estimate near to the true position) needed in the linear approximation in (15). A reasonable approximation of this first position estimate is the same estimate given by the associated subset $\mathbf{x}_{0k} = \widehat{\mathbf{x}}_k$ which is our proposal to obtain (13). Numerical simulations will show that this approximation has no impact in the estimator accuracy for all range of SNR.

The conclusion is that the general ML position estimate in an scenario with N independent TDOA measurements can be approximated as a linear combination of K partial ML position estimates. These are obtained as the minimizers of the negative log-likelihood function defined in (20) taking only into account the TDOA measurements contained in the associated subset. Note here, that these K subsets of TDOA measurements follow these rules:

- An original TDOA measurement is always included in, at least, one subset.
- An original TDOA measurement can be included in several subsets.
- The variance associated to each TDOA measurement included in a subset is the original variance of that TDOA measurement multiplied by the number of times that this TDOA measurements has been used. This is, the number of subsets that include this measurement (Q_n in (5), (6) and (7) for the n-th measurement).
- The number of subsets, the number of the elements inside each subset and the order of the elements inside are absolutely arbitrary.

4. ASSIGNMENT ALGORITHM

Taking into account the ML decomposition shown in the previous section, we can express the ML position estimation ($\widehat{\mathbf{x}}$) as a linear combination of several partial ML-estimations considering a subset of the original TDOA measurements.

The idea here consists in applying the closed-form Fang's estimator [5] to each subset, motivated by the fact that this is the ML estimator if the number of TDOA measurements included in the subset coincides with the dimension of the position to be estimated. So we have to divide the original N TDOA measurements in K subsets of length two (for the 2D case) following the rules shown in the previous section. Of course, there is a lot of possibilities to perform this partition.

The problem here is that Fang's estimator fails in certain scenarios. Concretely, it may give an ambiguity of two points (two solutions of $\widehat{\mathbf{x}}_F$) or may not give a solution at all. This can be solved in most cases by doing the partition explained before in a different way (different subsets). Now, we present a reasonable procedure to perform the partition avoiding the problems presented by the original Fang's estimator (See notation notes bellow).

1. Φ_N = set of indexes of the non-used TDOA measurements
2. Φ_U = set of indexes of the used TDOA measurements
3. **If** Φ_N is null $\rightarrow$ Finish assignation
4. **For** m = $\{\Phi_N - \Phi_N(1)\}$
5. subset = { $t_{\Phi_N(1)}$, t_m }
6. **If** Fang(subset)
7. WRITE subset
8. Go to 1
9. **End If**
10. **End For**
11. **For** m = $\{\Phi_U\}$
12. subset = { $t_{\Phi_N(1)}$, t_m }
13. If Fang(subset)
14. WRITE subset
15. Go to 1
16. End If
17. **End For**
18. Discard $t_{\Phi_N(1)}$
19. Go to 1

Notation notes: By *non-used TDOA measurement*, we mean all initial TDOA measurements not assigned yet to a subset. *Fang(subset)* is a function that returns TRUE if Fang's position estimator produces a non ambiguous solution using the TDOA measurements included in *subset*. Instruction *WRITE subset* means that this *subset* is selected to be used and t_m is the m-th original TDOA measurement. By definition, the instruction m = $\{\Phi\}$ means: for m = all values included in Φ. Finally the instruction $\Phi_N - \Phi_N(1)$ indicates the set of all elements of Φ_N except the first one. Note that in line 18, this non-used TDOA measurement is discarded because it is impossible to produce a position estimate with the Fang's algorithm. This means that the hyperbola associated to this TDOA measurement does not intersect with any other hyperbola of any other TDOA measurement. If this occurs with all N TDOA measurements, it is said that this is an outlier.

5. SIMULATION RESULTS

In this section, the performance analysis of the proposed algorithm is presented. The scenario used in the 2D location simulations presented, consists in a set of $L = \{3, 4, 5$ and $6\}$ BSs placed uniformly at a circumference of radius 1000 meters centered at the origin of coordinates. The mobile has been placed at the point [10, 10] (meters). Although the proposed algorithm accepts an arbitrary distribution of variances for the TDOA measurements, it has been simulated, for clarity, a common standard deviation $\sigma_k = \sigma, \forall k$ from

10 to 3000 meters in order to cover all possible scenarios (high and low SNR). The algorithm is implemented using the assignment algorithm strategy in section (4). The performances analyzed are the mean square error (M.S.E) and the outlier probability (probability of no valid solution) over 10000 trials.

In figure (1), the ratio between the M.S.E. and the CRB [6] is depicted for the original Fang's algorithm and for the proposed one. Note here that in scenarios with more than two TDOA measurements, the Fang algorithm is applied to the two TDOA measurements that produce the minimum position error. It can be clearly seen that the proposed algorithm attains the CRB at all reasonable SNR values. It can be also seen that the original Fang's algorithm only attains the CRB in the scenario with two TDOA measurements, this is, minimum number of TDOA measurements for 2D location(Fang's algorithm performance with two TDOA measurements equals the proposed algorithm performances).

In figure (2) it can be seen at the right side of the figure (using the right axis) the outlier probabilities for the simulated scenarios. Again, for reasonable values of σ, the outlier probability remains at zero. In the same figure (using the left axis), it can be seen the "effective number of TDOA measurements". By this concept we mean the mean number of TOAS used by the proposed algorithm. Note that this reduction in the number of TDOA measurements is produced by the discarding process performed in line 18 of the pseudo-code shown in the previous section. This figure shows that for hight enough values for σ, the information of the TDOA measurements collapses. Note that this phenomenon is produced for values of σ similar to the distance from the mobile to the BSs (1000m in this simulation), this is, for very unlikely SNR scenarios.

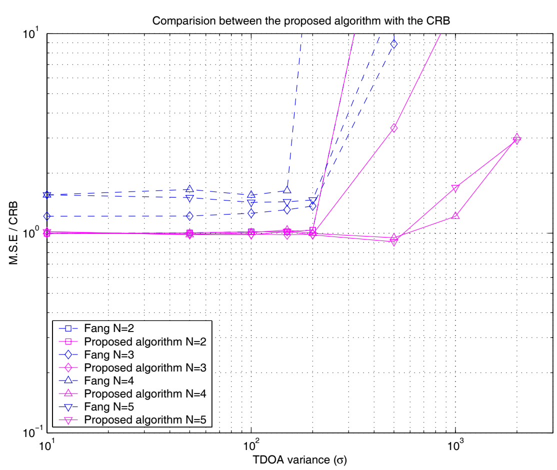

Fig. 1. M.S.E. normalized by the CRB

6. CONCLUSIONS

This paper has presented a generalized version of the Fang's algorithm [5] applicable to an arbitrary number of TDOA measurements are available. Some properties of the non-convex ML function of the problem have motivated to split

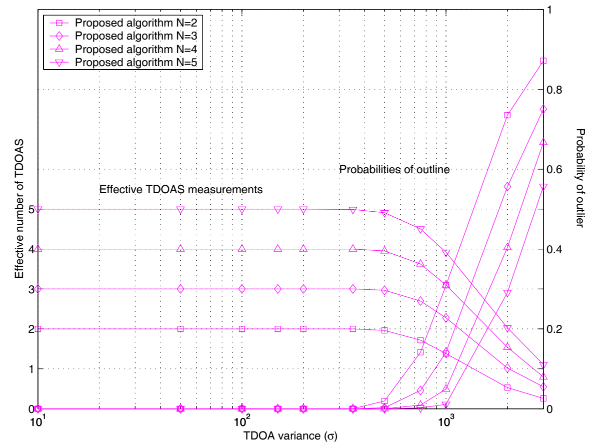

Fig. 2. Prob. of outlier and effective number of TDOAs

the general ML function minimization into several partial ML function minimizations where the original algorithm can be applied to.

Simulation results confirm that this partition of the general ML function maintains the asymptotic optimality and allows the implementation of a very low complexity extension of the Fang's algorithm.

A proposed method is presented to perform this partition taking into account the limitations of the original Fang's algorithm.

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