PERFORMANCE BREAKDOWN OF SUBSPACE-BASED METHODS IN ARBITRARY ANTENNA ARRAYS: GLRT-BASED PREDICTION AND CURE

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ABSTRACT

We continue to develop a technique based on the generalised likelihood-ratio test (GLRT) that is capable of identifying and rectifying direction-of-arrival (DOA) estimation outliers that arise with the use of any subspace-based DOA estimation technique. While our previously introduced likelihood ratio (LR) maximisation techniques relied upon specific properties of an assumed Toeplitz covariance matrix, in this paper we introduce a technique applicable to arbitrary antenna array geometries.

1. INTRODUCTION

All subspace-based DOA estimation methods suffer from a well-known rapid degradation in performance as either the signal-to-noise ratio (SNR) or the number of observed samples (snapshots) N drops below a certain value that is called the SNR or N threshold [1, 2]. A considerable amount of research has been conducted into such threshold conditions, which has determined that a DOA estimation outlier is caused by a so-called "subspace swap" [3], and more recent attempts to determine from the data whether or not a subspace swap has actually occurred.

In our previous papers [4, 5], we suggested a novel GLRT-based approach for subspace-specific outlier identification. This approach relies upon the straight-forward observation that the LR generated by the exact (true) covariance matrix appears as a random process whose p.d.f. does not depend on the signal scenario, but depends only on the number of antenna array elements (M) and the sample support N. We suggested that subspace-based DOA estimates should be augmented by corresponding power estimates, allowing us to form a covariance matrix model which is then used to compute the LR. Comparison of this LR with the threshold provided by the above scenario-free distribution allows us to identify potential outliers. This is because of another straight-forward observation that any truly ML estimates must generate a LR that exceeds that of the exact covariance matrix parameters. We demonstrated in [4, 5] that for linear antenna arrays, DOA estimation outliers that arise with the use of any subspace-based DOA estimation technique could be reliably determined, since they generate LRs that are significantly distinct from the scenario-free p.d.f. of the exact covariance matrix.

At the same time, we showed that if the sample support and/or SNR continue to decrease, then we eventually reach the regime where completely erroneous estimates may generate very high LR values. This final ML breakdown is inherent, and cannot be predicted nor cured within the ML paradigm. This demonstrates the existence of ML "performance breakdown" conditions, together with an important "gap" between these ML threshold conditions and those of any particular subspace-based DOA estimation algorithm.

Our proposed outlier mitigation technique ("cure") for uniform and for sparse linear arrays (that have a uniform co-array) [4, 5] heavily relies upon the Toeplitz properties of the underlying covariance matrix. This paper describes techniques for the general case of arbitrary antenna arrays, where this approach could not be used.

While some quite efficient methods for rectifying covariance matrix estimates for arbitrary geometry arrays have been introduced [6], unfortunately there is as yet no known existence condition (analogous to the famous Carathéodory theorem [7]) for arbitrary geometries, and so a LR-maximisation technique based on covariance matrix parameters (similar to LR optimisation over the class of Toeplitz covariance matrices [5]) is not applicable here. For this reason, we propose a new GLRT-based scheme for "outlier rectification" that can be used for any array geometry.

2. PROBLEM FORMULATION

Consider an array of M omnidirectional sensors whose steering vector is $s(\theta)$, where θ is a plane-wave signal's DOA. For simplicity of description, assume here that the array and sources are coplanar. Assuming $m < M$ uncorrelated Gaussian sources, we may express the vector of observed sensor outputs (the "snapshot") at time t as

$$\boldsymbol{y}(t) \;=\; S(\boldsymbol{\theta})\,\boldsymbol{x}(t) + \boldsymbol{\eta}(t) \qquad \text{for} \quad t = 1, \ldots, N \tag{1}$$

where $\boldsymbol{x}(t) \in \mathcal{C}^{m\times 1}$ are the Gaussian signal amplitudes with DOAs $\boldsymbol{\theta} \equiv [\theta_1, \ldots, \theta_m]^T$ and powers $P \equiv \operatorname{diag}[p_1, \ldots, p_m]$, the array-signal manifold matrix is $S(\boldsymbol{\theta}) \equiv [\boldsymbol{s}(\theta_1), \ldots, \boldsymbol{s}(\theta_m)] \in \mathcal{C}^{M\times m}$, and $\boldsymbol{\eta}(t) \in \mathcal{C}^{M\times 1}$ is Gaussian white noise of power p_0:

$$\boldsymbol{x}(t) \sim \mathcal{CN}(m,\, 0,\, P), \quad \boldsymbol{\eta}(t) \sim \mathcal{CN}(M,\, 0,\, p_0 I_M) \tag{2}$$

where $\mathcal{CN}(M,\, 0,\, R)$ denotes a complex (circular) Gaussian distribution of dimension M with zero mean and covariance matrix R.

Therefore the input data is described by the complex Gaussian distribution $\mathcal{CN}(M,\, 0,\, R)$, where

$$R \;=\; S(\boldsymbol{\theta})\, P\, S(\boldsymbol{\theta})^H + p_0\, I_M\,. \tag{3}$$

We assume that the snapshots are statistically independent:

$$\mathcal{E}\{\boldsymbol{y}(t_1)\,\boldsymbol{y}^H(t_2)\} \;=\; \begin{cases} R & \text{for} \quad t_1 = t_2 \\ 0 & \text{for} \quad t_1 \neq t_2\,, \end{cases} \tag{4}$$

and so the sufficient statistic for any inference regarding this data is the direct data covariance (DDC) (sample) matrix

$$\hat{R} \;=\; \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{y}(t)\,\boldsymbol{y}^H(t) \tag{5}$$

where $N\hat{R}$ is described by the complex Wishart distribution $\mathcal{CW}(N,\, M;\, R)$ [8].

In this paper, we concentrate on the performance-breakdown issues of subspace-based DOA estimation techniques, and therefore can assume that the number of sources m ($m < M$) is known, or has been estimated by some routine, such as the Wax–Kailath ITC technique [9]. (Moreover, in dealing with subspace-based techniques, we also assume that the conditions for identifiability have been met, and that the array-signal manifold matrix $S(\boldsymbol{\theta})$ is not column-rank deficient.) Thus the eigendecomposition of $\hat{R}$ is

$$\hat{R} \;=\; \hat{U}_s \hat{\Lambda}_s \hat{U}_s^H + \hat{U}_n \hat{\Lambda}_n \hat{U}_n^H \tag{6}$$

where $\hat{U}_s$ are the m signal-subspace eigenvectors; $\hat{\Lambda}_s \equiv \operatorname{diag}[\hat{\lambda}_1, \ldots, \hat{\lambda}_m]$ is the m-variate diagonal matrix of signal-subspace eigenvalues; while $\hat{U}_n$ and $\hat{\Lambda}_n$ are similarly the $(M-m)$-variate matrices relating to the noise-subspace. If we choose MUSIC as our particular DOA estimation technique, then the m DOA estimates $\hat{\boldsymbol{\theta}} \equiv [\hat{\theta}_1, \ldots, \hat{\theta}_m]$ are computed from the coordinates of the m largest maxima in the MUSIC pseudo-spectrum

$$f(\theta) \;=\; [\boldsymbol{s}^H(\theta)\,\hat{U}_n\,\hat{U}_n^H\,\boldsymbol{s}(\theta)]^{-1}\,. \tag{7}$$

Given the DOA estimates $\hat{\boldsymbol{\theta}}$, the corresponding power estimates $\hat{\boldsymbol{p}}$ are found in the traditional manner [10]:

$$\hat{P} \;=\; \operatorname{diag}_+\left\{\beta\, S^H(\hat{\boldsymbol{\theta}})\,[\hat{R} - \hat{p}_0 I_M]\, S(\hat{\boldsymbol{\theta}})\,\beta\right\} \tag{8}$$

where

$$\beta \equiv [S^H(\hat{\boldsymbol{\theta}})\, S(\hat{\boldsymbol{\theta}})]^{-1}\,, \quad \hat{p}_0 \;=\; \frac{1}{M-m}\sum_{j=1}^{M-m} \hat{\lambda}_{m+j} \tag{9}$$

and $\operatorname{diag}_+\{\cdot\}$ means take the non-negative diagonal elements to form the vector, while replacing the negative elements (if any) by zeros.

Given the DDC matrix $\hat{R}$ and the MUSIC-generated model matrix

$$\tilde{R} \equiv S(\hat{\boldsymbol{\theta}})\, \hat{P}\, S(\hat{\boldsymbol{\theta}})^H + \hat{p}_0\, I_M\,, \tag{10}$$

we need to decide whether the set of DOA estimates $\hat{\boldsymbol{\theta}}$ contains one or more outliers; identify which estimates are outliers (if any); and replace them by "proper" DOA estimates.

3. GLRT-BASED OUTLIER IDENTIFICATION

Given the sample covariance matrix $\hat{R}$ and the MUSIC-generated model matrix $\tilde{R}$, the sphericity test [11] for the hypothesis

$$\begin{aligned} H_0 :&\quad \mathcal{E}\left\{\tilde{R}^{-\frac{1}{2}}\,\hat{R}\,\tilde{R}^{-\frac{1}{2}}\right\} = c_0 I_M \qquad \text{against} \\ H_1 :&\quad \mathcal{E}\left\{\tilde{R}^{-\frac{1}{2}}\,\hat{R}\,\tilde{R}^{-\frac{1}{2}}\right\} \neq c_0 I_M\,, \qquad c_0 > 0 \end{aligned} \tag{11}$$

may be used to accept or reject this model. For Gaussian mixtures, the LR for this test is $\gamma(\tilde{R}) = \gamma_0^N(\tilde{R})$, where we have implicitly defined γ_0 as follows:

$$\gamma(\tilde{R}) \;=\; \gamma_0^N(\tilde{R}) \equiv \left(\frac{\det(\tilde{R}^{-1}\hat{R})}{\left[\frac{1}{M}\operatorname{tr}(\tilde{R}^{-1}\hat{R})\right]^M} \right)^N \tag{12}$$

($0 < \gamma(\tilde{R}) \leq 1$), with the set of estimated parameters $\{\hat{\boldsymbol{\theta}},\, \hat{\boldsymbol{p}}/\hat{p}_0\}_{ML}$ being those that yield the global maximum of the LR $\gamma(\tilde{R})$, which coincides with the global maximum of the (stochastic) likelihood function [4, 5].

Since the ML-generated covariance matrix $R_{ML}(\hat{\boldsymbol{\theta}},\, \hat{\boldsymbol{p}}/\hat{p}_0)$ belongs to the same admissible set as the true covariance matrix R, we obviously have

$$\gamma(R_{ML}) > \gamma(R)\,. \tag{13}$$

Therefore any set of parameters estimates $\{\hat{\boldsymbol{\theta}},\, \hat{\boldsymbol{p}},\, \hat{p}_0\}$ (including our MUSIC-generated ones) that are deemed to be sufficiently close to ML estimates must obey the similar condition

$$\gamma(\tilde{R}) \geq \gamma(R)\,. \tag{14}$$

However, when $\gamma(\tilde{R}) < \gamma(R)$, then the set of (MUSIC-generated, say) estimates is not statistically close enough to the ML estimates, and should be disregarded, since it contains one or more outliers.

Recall that in extreme circumstances, when erroneous (DOA) estimates generate high LR values that meet condition (14), it means that such estimates are "as good as" the true ones in terms of the ML or LR criteria, and so cannot be predicted nor cured within the ML paradigm.

Naturally, in practical applications the exact covariance matrix R is unknown, in which case the strict inequality in (14) is replaced by statistical thresholding that heavily exploits the fact that the p.d.f. for $\gamma(R)$ does not depend on R, but is exhaustively specified by the two parameters M and N. Indeed, in [5], we derived the following (exact) expression for the p.d.f. of $\gamma_0(R)$:

$$f(\gamma_0) = \delta(M,N)\, \gamma_0^{N-M} G_{M,M}^{M,0} \left(\gamma_0 \Big|_{0,1,\ldots,M-1}^{\frac{M^2-1}{M},\frac{M^2-2}{M},\ldots,\frac{M^2-M}{M}} \right) \tag{15}$$

where

$$\delta(M,N) = (2\pi)^{\frac{M-1}{2}} M^{\frac{1-2MN}{2}} \frac{\Gamma(MN)}{\prod_{j=1}^{M} \Gamma(N-j+1)} \tag{16}$$

and $G_{M,M}^{M,0}(\cdot)$ is Meijer's G-function [12]. The moments of this distribution are

$$\mathcal{E}\{\gamma_0^h\} = M^{Mh} \frac{\prod_{j=1}^{M} \Gamma(N-j+1+h)}{\prod_{j=1}^{M} \Gamma(N-j+1)} \frac{\Gamma(MN)}{\Gamma(M[N+h])}\,. \tag{17}$$

Thus a set of (MUSIC-derived, say) estimates $\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{p}}, \hat{p}_0\}$ that generates a covariance matrix $\tilde{R}$ is treated as "proper" (not containing outliers) if the threshold condition

$$\gamma_0(\tilde{R}) > \alpha \tag{18}$$

is satisfied, where

$$\int_0^{\alpha} f(\gamma_0)\, d\gamma_0 \;=\; \rho \tag{19}$$

and ρ is any desired probability of incorrect identification (where the true ML estimate is treated as containing an outlier).

This technique addresses the first part of our identification problem. While this part does not depend on the specific antenna array geometry, and so is essentially the same as in the linear array case [4, 5], the following steps must replace the Toeplitz matrix-related approach that was developed there.

First, we try to identify which particular DOA estimates are the outliers. Initially, we may use the MUSIC-generated DOA estimates to initialise a direct (unconstrained) LR maximisation over the set of $(2m+1)$ parameters $\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{p}}, \hat{p}_0\}$. In rare cases, where the MUSIC outlier is in the convex proximity of an appropriate LR maximum, we may expect to get a solution that satisfies the threshold condition (18), and therefore the new set of estimates would not contain an outlier. If this occurs, then the outlier "identification-and-cure" problem is immediately solved. Unfortunately, this desirable outcome is rare; mostly the direct LR maximisation converges to an inappropriate local extremum that does not pass the threshold condition (18).

Assume for the moment that the estimated parameter set contains at most one outlier. Let us exclude in turn each source from the model by defining

$$\tilde{R}_j \equiv \tilde{R} - \hat{p}_j\, \boldsymbol{s}(\hat{\theta}_j)\, \boldsymbol{s}^H(\hat{\theta}_j) \qquad (j = 1, \ldots, m) \tag{20}$$

then we propose computing the function

$$F_j \equiv \gamma_0(\tilde{R}_j) \equiv \frac{\det(\tilde{R}_j^{-1}\hat{R})}{\left[\frac{1}{M}\,\mathrm{tr}\,(\tilde{R}_j^{-1}\hat{R})\right]^M} \tag{21}$$

and then finding the maximum of F_j over $j = 1, \ldots, m$. In other words, any source whose exclusion from the model does not lead to a significant degradation in LR can be consider an outlier.

The rationale behind this approach is that an incorrect DOA estimate cannot contribute significantly to the LR, while the defect in LR (18) is due to the fact that the correct DOA estimate is missing in $\tilde{R}$. Hence, if excluded from the model, any erroneous source should not invoke a significant additional degradation in LR, compared with the original $\gamma(\tilde{R})$. On the contrary, if a correctly estimated source is removed from the model, then a significant LR degradation should be observed.

4. OUTLIER RECTIFICATION

To summarise, if the LR does not exceed the threshold condition, then we assume our parameter set contains an outlier. Having identified the most likely outlier by removing each in turn, the final step is to replace the suspected outlier by a proper (ML) estimate. The GLRT philosophy stimulated the following simple two-step rectification algorithm.

Let $\tilde{R}^{(-1)}$ be the MUSIC-generated covariance matrix with one outlier removed from the model. We introduce the MUSIC-like function

$$f^{(-1+1)}(\theta) \equiv \gamma_0(\tilde{R}^{(-1+1)}) \tag{22}$$

where

$$\tilde{R}^{(-1+1)} \equiv \tilde{R}^{(-1)} + \hat{p}(\theta)\, \boldsymbol{s}(\theta)\, \boldsymbol{s}^H(\theta) \tag{23}$$

where $\hat{p}(\theta)$ is the estimate of the additional source with azimuth θ calculated using (8). We then simply find the maximum of the function $f^{(-1+1)}(\theta)$ and treat the DOA estimate

$$\hat{\theta}^{(-1+1)} \equiv \arg\max_{\theta} f^{(-1+1)}(\theta) \tag{24}$$

as the rectified outlier. In other words, to the $(m-1)$ reduced set of DOAs, we search all azimuths to find the most likely additional DOA.

The second step is to use the new m set of DOAs and powers as initialisation for a further application of the LR maximisation routine.

If this final set of refined parameters $\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{p}}, \hat{p}_0\}$ exceeds the LR threshold, then it is accepted, otherwise it is assumed that the original set contained more than one outlier, and hence we repeat the outlier identification and outlier rectification routines to eliminate further outliers. On rare occasions, this cycle may repeat indefinitely (so far, we have encountered it with probability less than 10^{-5}), in which case the iterations are terminated at some maximum limit, and that set of parameter estimates is either disregarded, or other optimisation routine (such as random search) is applied in order to satisfy the LR lower bound.

A companion paper [13] discusses the application of this general method to the particular case of uniform circular antenna arrays, and analyses the results of Monte-Carlo simulations for DOA estimation in the difficult threshold region.

5. SUMMARY AND CONCLUSIONS

We have presented a new GLRT-based approach for the identification and rectification ("prediction and cure") of DOA estimation outliers that is independent of antenna array geometry but relates to any particular subspace-based DOA estimation technique. A set of DOA estimates, augmented by corresponding power estimates, form a model covariance matrix that is treated as containing a DOA outlier if it generates a LR that does not exceed a threshold value set by a scenario-free p.d.f. In turn, if necessary, one outlier is identified as being the one that least degrades the LR when it is excluded from the covariance matrix model. Rectification of the DOA outlier is accomplished by a MUSIC-like one-dimensional search technique that conditionally maximises the LR. This provides a new set of m DOAs and powers that initialises a further refinement by another application of the LR maximisation routine. This cycle is repeated until all outliers are rectified, as judged by the LR eventually exceeding the threshold value. As discussed in the companion paper, the efficiency of this technique is found to be high. This method could be generalized for partially correlated sources, by using a suitable Hermitian matrix in (8).

6. REFERENCES

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