SINGLE-SNAPSHOT ROBUST DIRECTION FINDING

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ABSTRACT

This paper presents a novel approach to estimate the directions of arrival recursively as measurements of incident signals are received along a sensor array. Using a single snapshot and without any statistical assumptions, the proposed method employs a robust performance criterion based on worst-case gain minimization. The performance of the new approach is evaluated by simulating the estimation algorithm for a linear array and comparing its performance to that of an existing single-snapshot algorithm.

1. INTRODUCTION

In the Direction-Of-Arrival (DOA) estimation problem, the directions of multiple incident signals on a sensor array are determined from measured data. This problem is of great importance in the fields of radar, sonar, and wireless communications. A practical direction-finding algorithm must provide a real-time solution that is computationally efficient and uses only a few array snapshots.

Most of the techniques that are suitable for real-time implementation have aimed at reducing the computational load of subspace decomposition per update but not the number of array snapshots necessary to attain a certain level of performance [1, 2]. At present, only a couple of single-snapshot algorithms have been reported in the literature using Bayesian and MuSiC-based approaches [3, 4]. At the same time, a practical DOA estimation method ought to be robust against imperfections in the array model that manifest themselves as amplitude and phase perturbations. Such unmodeled effects degrade the performance of subspace fitting algorithms [5].

This paper addresses the robustness issue by controlling, within a specified or minimum tolerance, the quotient of the cumulative estimation error by the sum of the cumulative perturbation and noise. On the premise of the worst possible perturbation that maximizes the quotient, a deterministic estimation approach arises, known as $\mathcal{H}_\infty$ filtering. The design of linear $\mathcal{H}_\infty$ filters has attracted significant attention in the literature [6]. In the nonlinear case, using a first-order approximation of the model at hand, an extended $\mathcal{H}_\infty$ filter exists locally and has a feasible design procedure [7, 8]. In the array signal processing field, one finds the linear $\mathcal{H}_\infty$ filter used in order to enhance the performance of the MuSiC algorithm against the effects of finite sampling and modeling error [9]. Beamforming is another area where linear $\mathcal{H}_\infty$ filtering has been employed to provide robustness when small samples or imperfect array models are used [10, 11].

Moreover, this paper proposes an extended $\mathcal{H}_\infty$ filter for DOA estimation that leads naturally to a single snapshot and computationally efficient algorithm. The $\mathcal{H}_\infty$ approach is deterministic and, therefore, it neither requires statistical knowledge of the modeling error nor makes any assumptions (e.g., zero-mean Gaussian) on the additive noise.

The model for the sensor array and a state-space realization of it are derived in Section 2. Section 3 formulates the parameter estimation as an extended $\mathcal{H}_\infty$ filter design problem and presents the result on the convergence of the parameter estimates. Simulation results for a linear sensor array are summarized in Section 4.

2. SENSOR ARRAY RESPONSE

The DOA of a plane wave sinusoidal signal is computed using a single snapshot of measurements from a uniformly-spaced, linear array (ULA). Note that the DOA ψ_i is measured from the broadside axis. The number of signals or sources is denoted by p and the number of sensors is denoted by J where $J \geq p$.

Given the sources $s_i(t) = e^{j\omega t}$, $i = 1, \ldots, p$, the measurement $y_l(t)$ at each sensor l takes the form

$$\sum_{i=1}^{p} s_i(t - \tau(\psi_i)\, l) = \sum_{i=1}^{p} e^{-j\phi(\psi_i)\, l}\, s_i(t) \quad (2.1)$$

where $\tau(\psi_i)$ is the delay and $\phi(\psi_i) := \omega\, \tau(\psi_i)$ is the

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phase lag. In the sequel, $t = 0$ is assumed because a single snapshot is used and the measurement at each sensor l is given by

$$y(l) = \sum_{i=1}^{p} e^{-j\phi(\psi_i)\, l} \qquad (2.2)$$

If d is the distance between sensors and c is the proroga-tion speed, the delay $\tau(\psi_i) := \frac{d}{c}\sin(\psi_i)$ is the elapsed time between when the signal is received at adjacent sensors.

The recursive solution of the $\mathcal{H}_\infty$ filtering problem requires the model to be in state-space form. The sensor array response in (2.2) can be written as a diagonal state equation

$$x_i(l+1) \;=\; e^{-j\,\phi(\psi_i)}\, x_i(l), \quad i = 1, \ldots, p \qquad (2.3)$$

Using (2.2), the output equation takes the form

$$y(l) \;=\; [\ 1 \ \ \cdots \ \ 1\ ]\, x(l) + w(l) \qquad (2.4)$$

where $w(l)$ incorporates the modeling uncertainty and the measurement noise.

The DOA parameters can be extracted from the above realization using the expression

$$\psi_i = \sin^{-1}\left(\frac{c\,\phi(\psi_i)}{\omega\, d}\right) \qquad (2.5)$$

after $\phi(\psi_i)$ is determined from each diagonal element of the state equation (2.3).

3. EXTENDED $\mathcal{H}_\infty$ FILTERING AND PARAMETER CONVERGENCE

The state-space realization of the array model, namely, equations (2.3) and (2.4), belong to the following class of stable state equations:

$$\begin{aligned} x(l+1) &= A(\theta)\, x(l),\ x(0) = x_0, \quad A \in \mathcal{C}^{p\times p} \\ y(l) &= C(\theta)\, x(l) + w(l), \quad C \in \mathcal{C}^{1\times p} \end{aligned} \qquad (3.6)$$

where w is some disturbance and $\mathcal{C}$ denotes the complex field. The vector θ comprises all of the model parameters that are unknown (e.g., $\theta_i = e^{-j\,\frac{d\,\omega\,\sin(\psi_i)}{c}}$). The objective is to identify the parameter vector, θ, using the measured data y where the initial state x_0 is unknown.

The parameter estimation problem is transformed into a state estimation problem by defining an augmented system with the state $z^T(l) = [x^T(l)\ \theta^T(l)]$. Since θ is independent of the sensor location, the evolution of θ along the array is described by $\theta(l+1) = \theta(l)$. As a result, the augmented system has a state equation

$$\begin{aligned} z(l+1) &= f(z(l)) \\ y(l) &= h(z(l)) + w(l) \\ \theta(l) &= L\, z(l) \end{aligned} \qquad (3.7)$$

where

$$\begin{aligned} f(z(l)) &:= \begin{bmatrix} A(\theta)\, x(l) \\ \theta(l) \end{bmatrix} \\ h(z(l)) &:= C(\theta)\, x(l) \\ L &:= \begin{bmatrix} 0_p & I_p \end{bmatrix} \end{aligned}$$

To estimate Lz, the following filter is proposed:

$$\begin{aligned} \hat{z}(l+1) &= f(\hat{z}(l)) + N(l)[y(l) - h(\hat{z}(l))] \\ \hat{\theta}(l) &= L\hat{z}(l) \end{aligned} \qquad (3.8)$$

where $N(l) \in \mathcal{C}^{2\,p\times 1}$ is the filter's gain. In the $\mathcal{H}_\infty$ filtering problem, the filter's gain, $N(l)$, determines the $\mathcal{L}_2$ gain from the sum of the disturbance $w(l)$ and the initial state estimation error $e_z(0) := \hat{z}(0) - z(0)$ to the estimation error $e_\theta(l) := \hat{\theta}(l) - \theta$. This approach protects against the worst-case disturbance and ensures that the $\mathcal{L}_2$ norm of the estimation error will be small if the disturbance and initial state estimation error norms are small. Given a tolerance γ, the suboptimal $\mathcal{H}_\infty$ problem is to find $N(l)$ so that the $\mathcal{L}_2$ gain is bounded as follows:

$$\sup_{w,\ e_z(0)} \frac{\|e_\theta\|^2_{2,[0,J]}}{\|w\|^2_{2,[0,J]} + e_z^*(0)\, R\, e_z(0)} < \gamma^2 \qquad (3.9)$$

where the positive-definite matrix $R \in \mathcal{C}^{2\,p\times 2\,p}$ is specified by the designer and the supremum is taken over all combinations w and $e_z(0)$ such that the denominator is not identically zero. In the sequel, the gain $N(l)$ is computed from the solution of (3.9) for the linearized system, which makes (3.8) an extended $\mathcal{H}_\infty$ filter. The idea of using the extended filter to estimate the parameters of a model has its origin in the theory of the Kalman filter [12].

The linear approximation of the augmented system (3.7) has the form

$$\begin{aligned} z(l+1) &= F(l)\, z(l) \\ y(l) &= H(l)\, z(l) + w(l) \\ \theta(l) &= L\, z(l) \end{aligned} \qquad (3.10)$$

where F and H the Jacobian matrices of the nonlinear mappings f and h, respectively. Using the realization of the linearized system (3.10), the extended $\mathcal{H}_\infty$ filter's gain is given by

$$N(l) = F(l)\, \bar{Q}(l)\, H^*(l)[H(l)\, \bar{Q}(l)\, H^*(l) + 1]^{-1} \qquad (3.11)$$

where $\bar{Q}(l) \in \mathcal{C}^{2p \times 2p}$ is related to the solution of the Riccati equation

$$\begin{aligned} Q(l+1) &= F(l)\,\bar{Q}(l)\,F^*(l) \\ &\quad -N(l)\,[H(l)\,\bar{Q}(l)\,H^*(l)+1]\,N^*(l) \\ \bar{Q}(l) &= (Q^{-1}(l)-\gamma^{-2}\,L^*\,L)^{-1} \\ Q(0) &= R^{-1} \end{aligned} \tag{3.12}$$

In the case that a projection feature is employed to keep the estimate, $\hat{\theta}(l)$, inside a compact subset of $\{\theta | A(\theta) \text{ is stable}\}$, the parameter update algorithm converges as stated below.

Proposition 1 Let the state-space equation (3.6) model the response of a ULA of J sensors to p incident signals. Suppose the extended $\mathcal{H}_\infty$ filter estimates the p-dimensional parameter vector, θ, which is equivalent to estimating the DOA of the p incident signals. Then, as the number of sensors, J, increases the estimate $\hat{\theta}(J)$ converges to a local minimum of the function

$$V(\theta) = \lim_{J\to\infty} \frac{1}{J} \sum_{l=1}^{J} |\bar{\varepsilon}(l,\theta)|^2$$

where

$$\begin{aligned} \bar{\hat{x}}(l+1,\theta) &= A(\theta)\,\bar{\hat{x}}(l,\theta) \\ \bar{\varepsilon}(l,\theta) &= y(l) - C(\theta)\,\bar{\hat{x}}(l,\theta) \end{aligned}$$

■

Note that the cost function, V, is independent of the filter gain, N, a property that has its origin in the absence of a constant term from the right-hand-side of the Riccati equation (3.12). The cost function, V, has been known to have this property also with respect to the extended Kalman filter, where a similar Riccati equation arises [13]. Therefore, both the extended Kalman and $\mathcal{H}_\infty$ filters converge to the set of the local minima of V.

4. SIMULATION RESULTS AND CONCLUSION

In this section, the $\mathcal{H}_\infty$ filter with a spatially-varying tolerance is applied to the DOA estimation problem for a ULA with half-wavelength spacing. The performance of the $\mathcal{H}_\infty$ filter is investigated for a number of cases to demonstrate its nominal performance and its robustness. The design parameters for the $\mathcal{H}_\infty$ filter are selected based on the analysis in [14].

For a single source ($p = 1$), the performance of the $\mathcal{H}_\infty$ filter is examined from ideal and noisy measurements. Furthermore, the robustness is investigated by considering gain and phase perturbations in the array model. The results are compared with an existing Bayesian single-snapshot algorithm [3] and demonstrate the superior performance of the $\mathcal{H}_\infty$ algorithm over standard, direction-finding methods. Specifically, the $\mathcal{H}_\infty$ algorithm delivers superior performance for noisy measurements with signal-to-noise ratio (SNR) less than $6dB$ and measurements generated by array models with gain, phase perturbations, or both. The two algorithms perform similarly for SNR $\geq 6dB$. For two sources ($p = 2$), the $\mathcal{H}_\infty$ filter's ability to separate targets is examined for ideal and noisy measurements.

4.1. Single Source: Measurement Noise

	SNR = 1.25 dB	
Algorithm	Mean (deg)	RMS Error (deg)
$\mathcal{H}_\infty$	40.1	1.42
Bayesian	64.5	45.7
MuSiC	40.0	N/A

Table 1: Estimation results with zero-mean, normally-distributed additive noise

To examine the performance of the $\mathcal{H}_\infty$ filter in a realistic situation, normally-distributed, zero-mean noise is added to the ideal measurements from 25 sensors. The mean estimate and the RMS error of the $\mathcal{H}_\infty$ and Bayesian algorithms are compared over 100 trials for several cases of the SNR and a true DOA of 40^o. For a SNR of $6dB$, the $\mathcal{H}_\infty$ and Bayesian algorithms estimate the DOA accurately with small RMS error. For smaller SNR, the $\mathcal{H}_\infty$ algorithm performs significantly better than the Bayesian algorithm. The mean and RMS error values for the two algorithms are shown in Table 1 for SNR values of $1.25dB$. Furthermore, the results of the $\mathcal{H}_\infty$ algorithm compare favorably to the estimate of the MuSiC algorithm using 100 snapshots. The results for a SNR of $0dB$ are similar.

4.2. Single Source: Perturbations

	$\mathcal{H}_\infty$		Bayesian	
Case	Mean	RMS Error	Mean	RMS Error
1	40.0	0.377	50.2	31.8
2	40.1	1.63	48.9	26.1
3	40.3	1.46	52.4	36.8
4	40.3	0.717	52.5	35.8

Table 2: Estimation results with gain and phase perturbations

To examine the robust performance of the $\mathcal{H}_\infty$ filter, the measurements are created using a perturbed array model and normally-distributed, zero-mean noise with $6dB$ SNR is added. As a result, the ideal array model used in the algorithm does not match the array model that generated the data. The latter is the perturbed model with gain perturbation and phase perturbation. As above, the mean estimate and the RMS error of the $\mathcal{H}_\infty$ and Bayesian algorithms are compared over 100 trials for a true DOA of 40^o. The four cases of gain and phase perturbation are (1) gain perturbation only, (2) phase perturbation only, (3) gain and phase perturbation simultaneously, and (4) 25 % sensor failure. The results of these four cases are summarized in Table 2 and demonstrate the superior robustness of $\mathcal{H}_\infty$ algorithm as compared to the Bayesian algorithm. Comparing the results in Table 1 (without uncertainty) and in Table 2 (with uncertainty), the performance of $\mathcal{H}_\infty$ algorithm remains nearly the same while the array uncertainty caused substantial degradation in the performance of the Bayesian algorithm.

4.3. Two sources

The aspects of the performance of the $\mathcal{H}_\infty$ filter specific to the dual-source ($p = 2$) case are examined from ideal and noisy measurements. Without measurement noise or model uncertainty, simulations results show that two sources can be identified using measurements from a reference sensor and two additional sensors. Furthermore, the $\mathcal{H}_\infty$ filter can estimate the directions of arrival for two targets whose directions are separated by a few degrees if the initial DOA estimates are sufficiently close to the true DOA values. When measurement noise and model uncertainty are absent, targets separated by 10^o can be distinguished if the DOA initial estimates are centered about the true directions and are separated by at most 40^o. For targets separated by 5^o, the DOA initial estimates can by separated by at most 15^o. For a SNR of $0dB$, the $\mathcal{H}_\infty$ algorithm can maintain 5^o separation but the initial DOA estimates can by separated by at most 10^o.

5. REFERENCES

[1] M. Moonen, F. J. Vanpoucke, and E. F. Deprettere, "Parallel and adaptive high-resolution direction finding," IEEE Transactions on Signal Processing, vol. 42, no. 9, pp. 2439–2448, 1994.

[2] D. J. Rabideau, "Fast, rank adaptive subspace tracking and applications," IEEE Transactions on Signal Processing, vol. 44, no. 9, pp. 2229–2244, 1996.

[3] B. M. Radich and K. M. Buckley, "Single-snapshot DOA estimation and source number detection," IEEE Signal Processing Letters, vol. 4, no. 4, pp. 109–111, 1997.

[4] R. K. Howell, "d-Music, a real-time algorithm for estimating the DOA of coherent sources using a single array snapshot," in Proceedings of the IEEE Conference on Acoustics, Speech, and Signal Processing, Phoenix, AZ, Mar. 1999, pp. 2881–2884.

[5] M. Viberg and A. L. Swindlehurst, "Analysis of the combined effects of finite samples and model errors on array processing performance," IEEE Transactions on Signal Processing, vol. 42, no. 11, pp. 3073–3083, 1994.

[6] U. Shaked and Y. Theodor, "$\mathcal{H}_\infty$-Optimal estimation: A tutorial," in Proceedings of the IEEE Conference on Decision and Control, Tuscon, AZ, Dec. 1992, pp. 2278–2286.

[7] U. Shaked and N. Berman, "H_∞ nonlinear filtering of discrete-time processes," IEEE Transactions on Signal Processing, vol. 43, no. 9, pp. 2205–2209, 1995.

[8] N. Berman and U. Shaked, "H_∞ nonlinear filtering," International Journal of Robust and Nonlinear Control, vol. 6, pp. 281–296, 1996.

[9] T. Ratnarajah and A. Manikas, "An $\mathcal{H}_\infty$ approach to mitigate the effects of array uncertainties on the MuSIC algorithm," IEEE Signal Processing Letters, vol. 5, no. 7, pp. 185–188, 1998.

[10] T. Ratnarajah and A. Manikas, "A robust signal-copy beamformer using $\mathcal{H}_\infty$ estimation," in Proceedings of the Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, CA, Nov. 1997, pp. 551–555.

[11] A.-C. Chang and C.-T. Chiang, "Adaptive $\mathcal{H}_\infty$ robust beamforming for imperfect antenna array," Signal Processing, vol. 82, pp. 1183–1188, 2002.

[12] A. H. Jazwinski, Stochastic process and filtering theory, vol. 64 of Mathematics in Science and Engineering, Academic Press, New York, 1970.

[13] L. Ljung, "Asymptotic behavior of the extended Kalman filter as a parameter estimator for linear systems," IEEE Transactions on Automatic Control, vol. 24, no. 1, pp. 36–50, 1979.

[14] Jr. R. T. O'Brien and K. Kiriakidis, "Single-snapshot robust direction finding," IEEE Transactions on Signal Processing, 2003, Submitted for publication.