ON 3-D HARMONIC RETRIEVAL FOR WIRELESS CHANNEL SOUNDING

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ABSTRACT

Multidimensional harmonic retrieval (HR) problems often appear in the context of MIMO wireless channel sounding. In particular, for a double-directional parametric MIMO channel model with uniform linear transmit and receive arrays, and a fixed wireless scenario (static - no Doppler), fitting the channel model parameters amounts to a 3-D harmonic retrieval problem. For this latter problem, we develop two new algorithms. One is based on conjugate-folding of the 3-D data and reduction to an eigenvalue decomposition problem; the other on a 3-D version of the Rank Reduction Estimator (RARE) applied to a subspace extracted from a single data snapshot, using 3-D conjugate-folding. Both algorithms remain operative close to the best known model identifiability boundary. The two algorithms are compared via pertinent simulations.

1. INTRODUCTION

Channel sounding campaigns are often undertaken by wireless developers and operators to characterize and model the MIMO wireless channel in a parsimonious way. In these campaigns, training data is emitted from the various transmit elements, and the associated data recorded from all receive elements is stored for subsequent channel estimation. Depending on the scenario, the associated channel parameter estimation problem often boils down to a multidimensional harmonic retrieval problem [1, 6, 4]. In particular, for the so-called *double-directional* MIMO wireless channel model, each path is modeled via five parameters: its direction-of-departure (DOD), direction-of-arrival (DOA), propagation delay, Doppler shift, and complex path loss. For uniform linear transmit and receive arrays and a fixed wireless scenario (zero Doppler shift), and after suitable preprocessing, the noiseless part of the baseband-equivalent data for a single MIMO snapshot can be modeled as [6, 4]

$$x_{k,l,m} = \sum_{f=1}^{F} d_f a_f^{k-1} b_f^{l-1} c_f^{m-1}, \qquad (1)$$

for $k = 1, \ldots, K$, $l = 1, \ldots, L$ and $m = 1, \ldots, M$, where K is the number of acquired data samples per channel, L is the number of receive antenna elements, M is the number of transmit antenna elements, $a_f := e^{-j\frac{2\pi}{K}\tau_f}$, $b_f := e^{-j\frac{2\pi d_R}{\lambda}cos\bar{\phi}_f}$, $c_f := e^{-j\frac{2\pi d_T}{\lambda}cos\bar{\theta}_f}$, and τ_f, $\bar{\phi}_f$, $\bar{\theta}_f$ are the delay, direction of arrival, and direction of departure, respectively, of the f-th path, λ is the carrier wavelength, and d_T, d_R are the inter-element spacings of the transmit and receive array, respectively. We see that extracting the parameters of interest is a 3-D harmonic retrieval problem: Given a mixture of F 3-D exponentials as in (1), find the parameter quadruples (a_f, b_f, c_f, d_f), for $f = 1, \ldots, F$. For ease of notation, we define $a_f := e^{j\omega_f}$, $b_f := e^{j\phi_f}$ and $c_f := e^{j\theta_f}$, and (1) yields

$$x_{k,l,m} = \sum_{f=1}^{F} d_f e^{j\omega_f(k-1)} e^{j\phi_f(l-1)} e^{j\theta_f(m-1)}, \qquad (2)$$

where $\omega_f, \phi_f, \theta_f \in \Pi$, and $\Pi := (-\pi, \pi]$. Define $\mathbf{X} \in \mathbb{C}^{K\times L\times M}$ with $\mathbf{X}(k,l,m) = x_{k,l,m}$, $\mathbf{A} \in \mathbb{C}^{K\times F}$ with $\mathbf{A}(k,f) = e^{j\omega_f(k-1)}$, $\mathbf{B} \in \mathbb{C}^{L\times F}$ with $\mathbf{B}(l,f) = e^{j\phi_f(l-1)}$, $\mathbf{C} \in \mathbb{C}^{M\times F}$ with $\mathbf{C}(m,f) = e^{j\theta_f(m-1)}$, and a diagonal matrix $\mathbf{D} \in \mathbb{C}^{F\times F}$ with $\mathbf{D}(f,f) = d_f$.

The following result will be used in our derivation.

Theorem 1 (*a.s. full rank of Khatri-Rao product of Vandermonde matrices* [2]) For a triple of Vandermonde matrices $\mathbf{A} \in \mathbb{C}^{K\times F}$, $\mathbf{B} \in \mathbb{C}^{L\times F}$ and $\mathbf{C} \in \mathbb{C}^{M\times F}$, with generators on the unit circle,

$$r_{\mathbf{A}\odot\mathbf{B}\odot\mathbf{C}} = \min(KLM, F), \quad P_{\mathcal{L}}(\mathbb{U}^{3F}) - a.s., \qquad (3)$$

where $\mathbb{U}$ is the unit circle, and $P_{\mathcal{L}}(\mathbb{U}^{3F})$ is the distribution used to draw the $3F$ generators for $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ assumed continuous with respect to the Lebesgue measure in $\mathbb{U}^{3F}$.

2. ALGORITHMS

Consider a sum of F 3-D undamped exponentials as in (2), with $k = 1, \ldots, K \geq 3$, $l = 1, \ldots, L \geq 3$, and $m = 1, \ldots, M \geq 3$, and assume

$$F \leq \left\lceil \frac{K}{2} \right\rceil \left\lceil \frac{L}{2} \right\rceil \left\lceil \frac{M}{2} \right\rceil. \qquad (4)$$

Define a 6-way array $\widehat{\mathbf{X}}$ with typical element

$$\begin{aligned}\widehat{x}_{k_1,k_2,l_1,l_2,m_1,m_2} &:= x_{k_1+k_2-1,l_1+l_2-1,m_1+m_2-1} \\ &= \sum_{f=1}^{F} d_f e^{j\omega_f(k_1-1)} e^{j\omega_f(k_2-1)} \\ &\quad e^{j\phi_f(l_1-1)} e^{j\phi_f(l_2-1)} e^{j\theta_f(m_1-1)} e^{j\theta_f(m_2-1)}, \qquad (5)\end{aligned}$$

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where $k_i = 1, \ldots, K_i \geq 2, l_i = 1, \ldots, L_i \geq 2, m_i = 1, \ldots, M_i \geq 2$ for $i = 1, 2$, with $K_1 + K_2 = K + 1$, $L_1 + L_2 = L + 1$, $M_1 + M_2 = M + 1$. Since $\min(K, L, M) \geq 3$ has been assumed, such extension to a 6-way array is always feasible. For $i = 1, 2$, define matrices

$$\mathbf{A}_i(k_i, f) := (e^{j\omega_f(k_i-1)}) \in \mathbb{C}^{K_i \times F},$$
$$\mathbf{B}_i(l_i, f) := (e^{j\phi_f(l_i-1)}) \in \mathbb{C}^{L_i \times F},$$
$$\mathbf{C}_i(m_i, f) := (e^{j\theta_f(m_i-1)}) \in \mathbb{C}^{M_i \times F}.$$

Then nest the six-way array $\widehat{\mathbf{X}}$ into a matrix $\widetilde{\mathbf{X}} \in \mathbb{C}^{K_1 L_1 M_1 \times K_2 L_2 M_2}$ as follows

$$\begin{aligned}\widetilde{x}_{p,q} &= \widehat{x}_{\lceil \frac{p}{L_1 M_1} \rceil, \lceil \frac{q}{L_2 M_2} \rceil, \lceil \frac{p}{M_1} \rceil - (\lceil \frac{p}{L_1 M_1} \rceil - 1) L_1,} \\ &\quad {}_{\lceil \frac{q}{M_2} \rceil - (\lceil \frac{q}{L_2 M_2} \rceil - 1) L_2, p - (\lceil \frac{p}{M_1} \rceil - 1) M_1, q - (\lceil \frac{q}{M_2} \rceil - 1) M_2} \\ &= \sum_{f=1}^{F} d_f g_{p,f} h_{q,f}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} g_{p,f} &:= e^{j\omega_f(\lceil \frac{p}{L_1 M_1} \rceil - 1)} e^{j\phi_f(\lceil \frac{p}{M_1} \rceil - (\lceil \frac{p}{L_1 M_1} \rceil - 1) L_1 - 1)} \\ &\quad e^{j\theta_f(p - (\lceil \frac{p}{M_1} \rceil - 1) M_1 - 1)}, \\ h_{q,f} &:= e^{j\omega_f(\lceil \frac{q}{L_2 M_2} \rceil - 1)} e^{j\phi_f(\lceil \frac{q}{M_2} \rceil - (\lceil \frac{q}{L_2 M_2} \rceil - 1) L_2 - 1)} \\ &\quad e^{j\theta_f(q - (\lceil \frac{q}{M_2} \rceil - 1) M_2 - 1)}, \end{aligned}$$

for $p = 1, \ldots, K_1 L_1 M_1$, and $q = 1, \ldots, K_2 L_2 M_2$. Define

$$\mathbf{G} := (g_{p,f}) \in \mathbb{C}^{K_1 L_1 M_1 \times F}, \qquad \mathbf{H} := (h_{q,f}) \in \mathbb{C}^{K_2 L_2 M_2 \times F}.$$

It can be verified that

$$\mathbf{G} = \mathbf{A}_1 \odot \mathbf{B}_1 \odot \mathbf{C}_1, \quad \mathbf{H} = \mathbf{A}_2 \odot \mathbf{B}_2 \odot \mathbf{C}_2. \tag{7}$$

Hence (6) can be written in compact matrix form as

$$\widetilde{\mathbf{X}} = \mathbf{G}\mathbf{D}\mathbf{H}^T. \tag{8}$$

Next, taking the conjugate of $x_{k,l,m}$ in (2), we obtain

$$x^*_{k,l,m} = \sum_{f=1}^{F} \widetilde{d}_f e^{j\omega_f(K-k)} e^{j\phi_f(L-l)} e^{j\theta_f(M-m)},$$

where $\widetilde{d}_f := d^*_f e^{-j\omega_f(K-1) - j\phi_f(L-1) - j\theta_f(M-1)}$. Define

$$\begin{aligned} y_{k,l,m} &:= x^*_{K-k+1, L-l+1, M-m+1} \\ &= \sum_{f=1}^{F} \widetilde{d}_f e^{j\omega_f(k-1)} e^{j\phi_f(l-1)} e^{j\theta_f(m-1)}, \end{aligned} \tag{9}$$

for $k = 1, \ldots, K$, $l = 1, \ldots, L$, $m = 1, \ldots, M$ and correspondingly the three-way array $\mathbf{Y} := (y_{k,l,m}) \in \mathbb{C}^{K \times L \times M}$. Following the same procedure as in the construction of $\widetilde{\mathbf{X}}$ from $\mathbf{X}$, we can construct a matrix $\widetilde{\mathbf{Y}} \in \mathbb{C}^{K_1 L_1 M_1 \times K_2 L_2 M_2}$ from $\mathbf{Y}$, such that

$$\widetilde{y}_{p,q} = \sum_{f=1}^{F} \widetilde{d}_f g_{p,f} h_{q,f}, \quad \text{i.e., } \widetilde{\mathbf{Y}} = \mathbf{G}\widetilde{\mathbf{D}}\mathbf{H}^T, \tag{10}$$

where $\widetilde{\mathbf{D}} = \mathrm{diag}(\widetilde{d}_1, \ldots, \widetilde{d}_F)$.

2.1. 3-D Multi-Dimensional (conjugate) Folding (3-D MDF)

Invoking Theorem 1, if $K_1 L_1 M_1 \geq F$ and $K_2 L_2 M_2 \geq F$, then both $\mathbf{G}$ and $\mathbf{H}$ in (7) are almost surely full column rank. Hence $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$ are full column rank, and the singular value decomposition of the stacked data yields

$$\begin{bmatrix} \widetilde{\mathbf{X}} \\ \widetilde{\mathbf{Y}} \end{bmatrix} = \begin{bmatrix} \mathbf{G}\mathbf{D} \\ \mathbf{G}\widetilde{\mathbf{D}} \end{bmatrix} \mathbf{H}^T = \mathbf{U}_{2K_1 L_1 M_1 \times F} \mathbf{\Sigma}_{F \times F} \mathbf{V}^H_{K_2 L_2 M_2 \times F}, \tag{11}$$

where $\mathbf{U}$ has F columns which together span the column space of $\left[\widetilde{\mathbf{X}}^T \ \widetilde{\mathbf{Y}}^T\right]^T$. Since the same space is spanned by the columns of $\left[(\mathbf{G}\mathbf{D})^T \ (\mathbf{G}\widetilde{\mathbf{D}})^T\right]^T$, there exists an $F \times F$ nonsingular matrix $\mathbf{T}$ such that

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{G}\mathbf{D} \\ \mathbf{G}\widetilde{\mathbf{D}} \end{bmatrix} \mathbf{T}. \tag{12}$$

It then follows that

$$\mathbf{U}_1^{\dagger} \mathbf{U}_2 = \mathbf{T}^{-1} \mathbf{D}^{-1} \widetilde{\mathbf{D}} \mathbf{T}, \tag{13}$$

which is an eigenvalue decomposition problem. $\mathbf{T}^{-1}$ contains the eigenvectors of $\mathbf{U}_1^{\dagger}\mathbf{U}_2$ (scaled to unit norm). Other parameters are given by

$$\mathbf{G}\mathbf{D} = \mathbf{U}_1 \mathbf{T}^{-1}, \qquad \mathbf{H} = \left[(\mathbf{G}\mathbf{D})^{\dagger} \widetilde{\mathbf{X}}\right]^T. \tag{14}$$

The first row of the product $\mathbf{G}\mathbf{D}$ is the diagonal of $\mathbf{D}$, i.e., $[d_1, \ldots, d_F]$ with its entries multiplied by a complex constant, i.e., $[(1/s_1)d_1, \ldots, (1/s_F)d_F]$. $(\omega_f, \phi_f, \theta_f)$ can be recovered from $\mathbf{G}$ and/or $\mathbf{H}$, e.g., the second, (M_2+1)-th and $(L_2 M_2 + 1)$-th rows of $\mathbf{H}$ are $[s_1 e^{j\theta_1}, \ldots, s_F e^{j\theta_F}]$, $[s_1 e^{j\phi_1}, \ldots, s_F e^{j\phi_F}]$, and $[s_1 e^{j\omega_1}, \ldots, s_F e^{j\omega_F}]$, respectively. All entries of a given column of $\mathbf{H}$ are multiplied by the same scaling constant. A simple way to determine $e^{j\theta_f}$, for example, is to evaluate the ratio of the $(2, f)$-th entry of $\mathbf{H}$ over the $(1, f)$-th entry of $\mathbf{H}$. $e^{j\phi_f}$ and $e^{j\omega_f}$ can be determined in a similar fashion. No pairing issue exists, i.e., $(\omega_f, \phi_f, \theta_f, d_f)$ are paired up automatically. Hence the parameter quadruples $(\omega_f, \phi_f, \theta_f, d_f)$, $f = 1, \ldots, F$ can be uniquely recovered almost surely, provided there exist integers K_1, K_2, L_1, L_2, M_1, M_2 such that

$$K_1 L_1 M_1 \geq F, \qquad K_2 L_2 M_2 \geq F, \tag{15}$$

subject to

$$K_1 + K_2 = K + 1, \quad L_1 + L_2 = L + 1, \quad M_1 + M_2 = M + 1. \tag{16}$$

If the integers are chosen such that

$$\begin{cases} \text{if } K \text{ is odd, pick } K_1 = K_2 = \frac{K+1}{2}, \\ \text{if } K \text{ is even, pick } K_1 = \frac{K}{2},\ K_2 = \frac{K+2}{2}, \end{cases} \tag{17}$$

and similarly for L_1, L_2, M_1 and M_2 then condition (16) is satisfied. Once we pick six integers following the above rules, condition (4) assures that inequality (15) holds for those particular integers. Given noisy observations, the procedure for estimating $(\omega_f, \phi_f, \theta_f, d_f)$, $f = 1, \ldots, F$, by the 3-D MDF algorithm is summarized in Table 1. We use $(\cdot)'$ to denote the noisy counterpart of $(\cdot)$. Notice that at step 4) in Table 1, due to the rich structure of the Khatri-Rao product of Vandermonde matrices, there are many ways to derive estimates of $(\omega_f, \phi_f, \theta_f)$ from $\mathbf{G}'$ and $\mathbf{H}'$; these may be combined, e.g., via simple averaging.

Table 1. The 3-D MDF algorithm

1) Form $\widetilde{\mathbf{X}}'$ and $\widetilde{\mathbf{Y}}'$ from $\mathbf{X}'$ using (5), (6), (9), and (10). The integers $K_1,\ K_2,\ L_1,\ L_2,\ M_1$, and M_2 are chosen according to (17).
2) Compute the F principal left singular vectors (i.e., $\mathbf{U}'$) of $[\widetilde{\mathbf{X}}'^T\ \widetilde{\mathbf{Y}}'^T]^T$. Partition $\mathbf{U}'$ into $\mathbf{U}'_1$ and $\mathbf{U}'_2$ as in (12).
3) Compute the eigenvectors of $\mathbf{U}_1'^{\dagger}\mathbf{U}'_2$, that is $(\mathbf{T}^{-1})'$ in (13). Obtain $\mathbf{G}'\mathbf{D}'$ and $\mathbf{H}'$ using (14).
4) The first row of $\mathbf{G}'\mathbf{D}'$ is $[c_1 d'_1, \ldots, c_F d'_F]$, the second, (M_2+1)th and $(L_2 M_2+1)$th rows of $\mathbf{H}$ are $[c_1 e^{j\theta'_1}, \ldots, c_F e^{j\theta'_F}]$, $[c_1 e^{j\phi'_1}, \ldots, c_F e^{j\phi'_F}]$ and $[c_1 e^{j\omega'_1}, \ldots, c_F e^{j\omega'_F}]$ respectively. $(\omega'_f, \phi'_f, \theta'_f, d'_f)$ are paired up automatically.

A simple way of estimating, e.g., θ_f, would be the following. Pick up the f-th column of $\mathbf{H}$ and evaluate the element-wise ratios of the second, third, ... ,M_2-th row over the first, second, ... ,(M_2-1)-th row respectively. Following the same procedure for the remaining (K_2L_2-1) groups of M_2 rows of the f-th column of $\mathbf{H}$, we obtain a collection of point estimates for $e^{j\theta_f}$. The same procedure can be carried out for the f-th column of $\mathbf{G}$, yielding another collection of point estimates; all these point estimates are averaged to obtain the estimate of $e^{j\theta_f}$. The estimates of (ω_f, ϕ_f) can be derived in a similar manner. The above process is the one used in the simulations.

2.2. Single-Snapshot 3-D RARE

3-D RARE [4] is one of the best available multidimensional harmonic retrieval techniques, surpassing earlier techniques (e.g., [1], as shown in [4]), and remaining close to the Cramér-Rao Bound once SNR is beyond a certain threshold. RARE was originally proposed for application to a signal subspace extracted from a covariance matrix. 3-D RARE exploits the fact that the said subspace is spanned by a double Khatri-Rao product of three Vandermonde matrices.

Estimation of the covariance matrix requires multiple snapshots, i.e., realizations of the 3-D harmonic mixture with different complex amplitudes. Alternatively, for a single harmonic mixture (snapshot), a suitable subspace can be extracted directly from (11). Notice that $\mathbf{V}$ in (11) has the required structure (double Khatri-Rao product of three Vandermonde matrices) for the application of 3-D RARE. This mode also exploits 3-D conjugate folding to overdetermine the problem, just like 3-D MDF. However, in 3-D RARE and in 3-D MDF two fundamentally different structural properties of the signal subspaces in (11) are exploited for extracting the signal parameters. For this reason, in 3-D RARE the subspace dimensions are restricted to $2K_1L_1M_1 \geq F$ and $K_2L_2M_2 \geq F$ rather than as in (17).

3. SIMULATION RESULTS

We report the results of two simulation experiments, for $6\times6\times6$ data and $F=3$ (3-D) harmonics in both cases.

The data is generated according to (2), and complex circular Gaussian white noise (in all three dimensions) is added. The frequencies are held fixed throughout the experiments:

$$\begin{aligned}(\omega_1,\omega_2,\omega_3) &= (0.55\pi, 0.41\pi, 0.34\pi)\,,\\ (\phi_1,\phi_2,\phi_3) &= (0.719\pi, 0.777\pi, 0.906\pi)\,,\\ (\theta_1,\theta_2,\theta_3) &= (0.906\pi, 0.276\pi, 0.358\pi)\,.\end{aligned}$$

In the first experiment, new amplitudes are generated for each run, according to an i.i.d. circular Gaussian distribution. In the second one they are generated once and remain fixed throughout.

The number of Monte-Carlo runs is 5×10^3 for the first experiment, and 2×10^3 for the second one. SNR is calculated by averaging

$$\mathrm{SNR} = 10\log_{10}\frac{\|\mathbf{X}\|^2}{KLM\sigma^2},$$

where σ^2 is the variance of the complex white noise. For 3-D MDF, the parameters K_1, K_2, L_1, L_2, M_1, M_2 were chosen according to (17); for 3-D RARE, the associated parameters were $K_1=L_1=M_1=3$, $K_2=L_2=M_2=4$. In Fig. 1 and Fig. 2 we plot the root mean square error (RMSE) of the MDF and RARE algorithms for the estimation of the aforementioned angular frequencies for the first and the second experiment respectively.

The two algorithms provide comparable results. Based on numerous experiments, our experience is that 3-D MDF tends to perform somewhat better when operating closer to the identifiability bound, whereas 3-D RARE has an edge when the problem is heavily overdetermined.

4. CONCLUSIONS

We have considered the 3-D harmonic retrieval problem in the context of double-directional MIMO wireless channel sounding. Two new algorithms were developed, based on 3-D conjugate folding. One stems from identifiability considerations, as in [3]; the other from covariance-domain 3-D RARE, but using 3-D conjugate folding to bypass the need for multiple snapshots. Simulations indicate that these algorithms are competitive performance-wise.

We are currently working towards the analysis of real measurement campaign data, courtesy of FTW.

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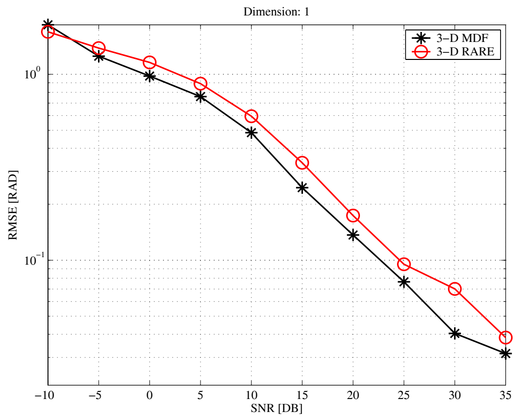
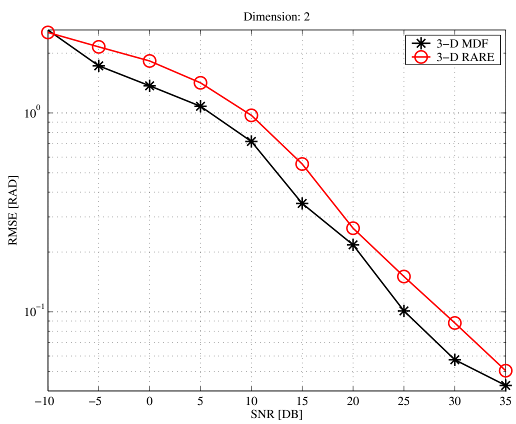
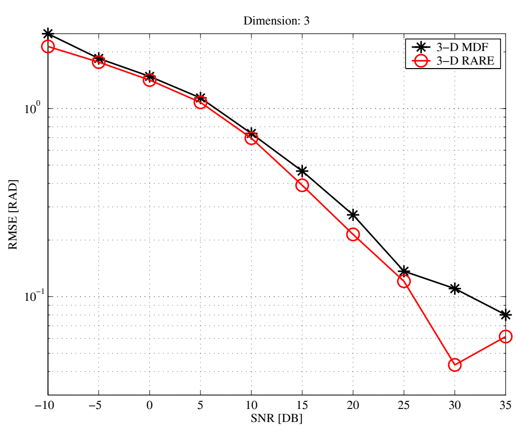


(a) Estimation of $\omega_1, \omega_2, \omega_3$.

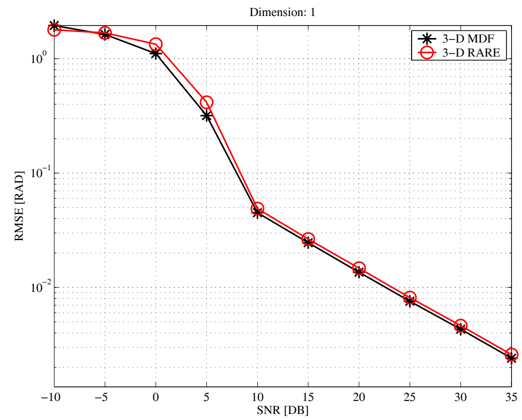


(b) Estimation of ϕ_1, ϕ_2, ϕ_3.

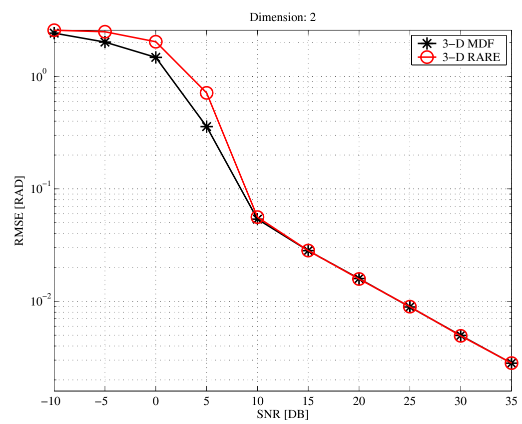


(a) Estimation of $\theta_1, \theta_2, \theta_3$.

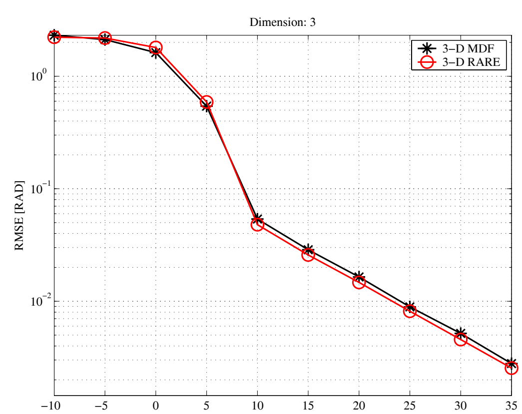
Fig. 1. $6 \times 6 \times 6$ array, i.i.d. circular complex Gaussian amplitudes.



(a) Estimation of $\omega_1, \omega_2, \omega_3$.



(b) Estimation of ϕ_1, ϕ_2, ϕ_3.



(a) Estimation of $\theta_1, \theta_2, \theta_3$.

Fig. 2. $6 \times 6 \times 6$ array, fixed complex amplitudes.