HIGH RESOLUTION ESTIMATION OF DIRECTIONS OF ARRIVAL IN NONUNIFORM NOISE

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ABSTRACT

We consider the problem of direction of arrival estimation in the presence of unknown spatially nonuniform noise. A subspace separation approach is applied, based on successive array element elimination, to isolate the contribution of the noise powers. A high resolution estimator based on MUSIC is described for nonuniform noise. Performance of the estimator is assessed through simulations and is compared to the Cramér-Rao Bound.

1. INTRODUCTION

Plane wave Direction Of Arrival (DOA) estimation using sensor arrays with a known geometry, such as Uniform Circular Arrays (UCA) or Uniform Linear Arrays (ULA), has received considerable attention in the literature [1]-[5]. The main estimation techniques which have been proposed are classified as eigen-decomposition and subspace fitting methods [2]-[4], or Maximum Likelihood (ML) techniques [1, 5].

High resolution methods such as MUSIC [2] are attractive due to the fact that under some conditions, they have the ability to surpass the limits related to classical Fourier-based methods [4]. Generally these methods rely on the partition of the observation space into a signal and noise subspaces. When the common assumption of spatially uniform noise is verified, it is possible to fully exploit the information embedded in the eigenvalues of the covariance matrix of data to perform subspace separation. When the noise is not uniform, i.e., when the noise powers are different from one sensor to another, it is not possible to use the eigenvalues of interest, unless the Signal to Noise Ratio (SNR) is extremely high and the angular separation between the DOAs is large enough. A number of cases where nonuniform noise is observed are described in [5] and a model highlighting its structure was addressed for the problem of DOA estimation, where the number of sources is known in advance. An efficient method based on ML was suggested. However, the method involves a highly nonlinear cost function and iterative optimization routines to bypass direct identification of the signal or noise subspaces.

In what follows, we propose a subspace separation method for a spatially nonuniform noise environment. The method copes with the spatial non-uniformity of the noise by successively eliminating the contribution of single elements from the array. A computationally more attractive high resolution estimator, Non-Uniform MU-SIC (NU-MUSIC) is proposed where the cost function reduces to a 1-D search over the DOA range.

2. DATA MODEL

Consider an array of M sensors receiving P narrow-band signals from sources with unknown DOAs, $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_P]^T$, where $(.)^T$ stands for matrix transpose. The sources are assumed to be coplanar and located in the far field. The number of sources P is assumed to be known and satisfy P < M.

The received signal vector at instant i can be modeled as [1, 5]

$$\mathbf{x}(i) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(i) + \mathbf{n}(i), \quad i = 1, \dots, L$$
(1)

where

$$\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \dots, \mathbf{a}(\theta_P)]$$
(2)

is the $(M \times P)$ -dimensional steering matrix and $\mathbf{a}(\theta_p), p=1,\ldots,P$ are the vectors of the array response to the directions $\boldsymbol{\theta}$. The functional form of vectors $\mathbf{a}(\theta_p)$ is assumed known. $\mathbf{s}(i)$ is the *P*-dimensional vector of the source signals and $\mathbf{n}(i)$ is the *M*dimensional vector of white sensor noise.

The additive noise $\mathbf{n}(i)$ is assumed to be a zero-mean spatially and temporally white Gaussian process with an unknown diagonal covariance matrix \mathbf{Q} , i.e.,

$$\mathbf{Q} = \mathsf{E}\left\{\mathbf{n}(i)\mathbf{n}^{H}(i)\right\} = \operatorname{diag}\left\{\mathbf{q}\right\}$$
(3)

where $(.)^{H}$ denotes Hermitian transpose and E(.) expectation. Spatial non-uniformity of the noise translates to different powers from one sensor to another, such that

$$\mathbf{q} = [\sigma_1^2, \sigma_2^2, \dots, \sigma_M^2]^T \tag{4}$$

The source signals and the noise are assumed to be uncorrelated. The array covariance matrix is therefore given by

$$\mathbf{R} = \mathsf{E}\left\{\mathbf{x}(i)\mathbf{x}^{H}(i)\right\} = \mathbf{A}(\boldsymbol{\theta})\mathbf{R}_{s}\mathbf{A}^{H}(\boldsymbol{\theta}) + \mathbf{Q}$$
(5)

where $\mathbf{R}_{\mathbf{s}} = \mathsf{E} \{\mathbf{s}(i)\mathbf{s}^{H}(i)\}\$ is the source signal covariance matrix. The received signal waveforms may be regarded as a random zero-mean Gaussian process [5, 6], satisfying the following

$$\mathbf{x}(i) \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$$
 (6)

Alternatively, if the received signals are assumed to be deterministic and unknown, they satisfy [5, 6]

$$\mathbf{x}(i) \sim \mathcal{N}(\mathbf{A}(\boldsymbol{\theta})\mathbf{s}(i), \mathbf{Q})$$
 (7)

3. SUBSPACE SEPARATION APPROACH

As the noise powers are different from one sensor to another, we alleviate the effect of noise due to the individual sensors by removing one element from the array.

For simplicity and without loss of generality, assume that we discard the 1-st element of the array. Thus, a reduced $(M-1) \times P$ dimensional steering matrix is obtained as follows

$$\mathbf{A}_1(\boldsymbol{\theta}) = [\mathbf{a}_1(\theta_1), \mathbf{a}_1(\theta_2), \dots, \mathbf{a}_1(\theta_p)]$$
(8)

where the vectors $\mathbf{a}_1(\theta_p)$ are the same as in (2) with the 1-st element removed. Similarly to (5), the covariance matrix of the collected data over the reduced (M-1)-element array is given by

$$\mathbf{R}_1 = \mathbf{A}_1(\boldsymbol{\theta}) \mathbf{R}_{\mathbf{s}} \mathbf{A}_1^H(\boldsymbol{\theta}) + \mathbf{Q}_1$$
(9)

where the reduced noise covariance matrix Q_1 is defined as

$$\mathbf{Q}_1 = \operatorname{diag}\left\{\mathbf{q}_1\right\} \tag{10}$$

with $\mathbf{q}_1 = [\sigma_2^2, \dots, \sigma_M^2]^T$. Observe that \mathbf{R}_1 is a sub-matrix of \mathbf{R} , i.e.,

$$\mathbf{R} = \begin{bmatrix} r_{11} & \mathbf{r}^H \\ \mathbf{r} & \mathbf{R}_1 \end{bmatrix}$$
(11)

where r_{11} is the (1, 1)-th element of **R** and vector **r** is defined as

$$\mathbf{r} = \mathbf{A}_1(\boldsymbol{\theta}) \mathbf{R}_{\mathbf{s}} \mathbf{b}_1^H \tag{12}$$

with \mathbf{b}_1 being the removed 1-st row of $\mathbf{A}(\boldsymbol{\theta})$.

Applying eigen-decomposition to the reduced matrix \mathbf{R}_1 results in ъ $\mathbf{D}\mathbf{D}\mathbf{D}^H$ (13)

$$\mathbf{R}_1 = \mathbf{E}\mathbf{D}\mathbf{E}^{\mathbf{T}}$$

$$\mathbf{E} = [\mathbf{e}_1, \dots, \mathbf{e}_{M-1}] \tag{14}$$

$$\mathbf{D} = \operatorname{diag}\left\{\lambda_1, \dots, \lambda_{M-1}\right\}$$
(15)

where $\lambda_m, m = 1, \ldots, M-1$, are the eigenvalues of \mathbf{R}_1 , and \mathbf{e}_m are their corresponding eigenvectors.

Using \mathbf{E} of (14), we define a matrix \mathbf{U} as follows

$$\mathbf{U} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{E} \end{bmatrix}$$
(16)

where **0** is an (M - 1)-dimensional vector of zero elements. It is straightforward to verify that U is unitary since it satisfies $UU^{H} =$ I, where I denotes the identity matrix. The resulting unitary transformation of the data covariance matrix is similar to the one introduced in [7]. Applying the unitary transformation U to the covariance matrix R leads to

$$\mathcal{R} = \mathbf{U}^{H}\mathbf{R}\mathbf{U}$$
(17)
$$= \begin{bmatrix} r_{11} & \mathbf{r}^{H}\mathbf{E} \\ \mathbf{E}^{H}\mathbf{r} & \mathbf{E}^{H}\mathbf{R}_{M}\mathbf{E} \end{bmatrix}$$
$$= \begin{bmatrix} r_{11} & \mathbf{c}^{H} \\ \mathbf{c} & \mathbf{D} \end{bmatrix}$$
(18)

Let $|c_1| \geq |c_2| \geq \ldots \geq |c_{M-1}|$ be the ordered magnitudes of the elements of vector c in (18). From (12), note that the *m*-th element c_m , has the following structure

$$c_m = \mathbf{e}_m^H \mathbf{A}_1(\boldsymbol{\theta}) \mathbf{R}_{\mathbf{s}} \mathbf{b}_1^H \tag{19}$$

The elements c_m of relation (19) can be interpreted as the projection of the 1-st column of \mathbf{R} onto the *m*-th eigenvector, \mathbf{e}_m , of \mathbf{R}_1 . Moreover, due to the fact that the noise subspace is orthogonal to the direction matrix $A_1(\theta)$, it is easy to verify that elements c_m satisfy the following

$$c_m \begin{cases} = 0, & \text{if } \mathbf{e}_m \text{ is a noise eigenvalue.} \\ \neq 0, & \text{if } \mathbf{e}_m \text{ is a signal eigenvalue.} \end{cases}$$
(20)

From a geometrical perspective, the first M - 1 eigenvalues of the transformed covariance matrix \mathcal{R} are the centers of M-1 Gerschgorin disks on the complex plane [7]. The magnitude of the Gerschgorin radii indicates the multiplicity of the eigenvalues and the subspaces that their eigenvectors span [7, 9]. In the case of the transformed covariance matrix \mathcal{R} of equation (17), using Gerschgorin's theorem [9], the radii of these disks, ρ_m , $m = 1, \ldots, M - 1$, are shown to be the magnitude of the elements c_m , i.e., $\rho_m = |c_m|$. Two distinct subsets of Gerschgorin disks can be easily identified. The first subset corresponds to the signal subspace for the first p radii, $|c_1|, \ldots, |c_P|$, and the second one corresponds to the noise subspace for the smallest and equal M-1-P radii, $|c_{P+1}|, \ldots, |c_{M-1}|$. Thus, based on the information contained in the elements c_m , $m = 1, \ldots, M - 1$, it is possible to separate the noise and signal subspaces.

4. NON-UNIFORM MUSIC ESTIMATOR (NU-MUSIC)

Because the noise is not spatially uniform, it is not possible to order the eigenvalues of R to separate the signal and the noise subspaces. Instead we use the information provided by the elements of vector \mathbf{c} as it is defined in (18), since they satisfy the following

$$|c_1| \ge |c_2| \ge \ldots \ge |c_P| \ge |c_{P+1}| = |c_{P+2}| = \ldots = |c_{M-1}| = 0$$
 (21)

When a criterion on the separation between the signal and noise subspaces is available, it is possible to use directly subspaceseparation-based estimators to retrieve the DOAs θ from the reduced covariance matrix \mathbf{R}_1 .

Let V be a matrix whose columns are the M - 1 - P eigenvalues of \mathbf{R}_1 , corresponding to the smallest M - 1 - P elements $|c_m|$ (or equivalently, Gerschgorin radii). In other words,

$$\mathbf{V} = [\mathbf{e}_{P+1}, \dots, \mathbf{e}_{M-1}] \tag{22}$$

As V spans the noise subspace, it is orthogonal to the direction vectors $\mathbf{a}_1(\theta_p), p = 1, \dots, P$. Thus a Non-Uniform MUSIC (NU-MUSIC) spectrum can be defined as follows

$$S_1(\theta) = \frac{1}{\mathbf{a}_1^H(\theta) \mathbf{V} \mathbf{V}^H \mathbf{a}_1(\theta)}$$
(23)

where subscript 1 represents the index of the removed element of **R** in forming **R**₁. The cost function $S_1(\theta)$ provides a solution over the range of θ . The estimates of θ are obtained as

$$\hat{\theta} = \arg\max_{\theta} \mathcal{S}_1(\theta) \tag{24}$$

Since the noise powers are not equal from one array element to another, accuracy of the spectrum $S_1(\theta)$ depends on the index of the particular array element to be removed. It is clear that M distinct NU-MUSIC spectra, $S_m(\theta), m = 1, \ldots, M$, can be obtained



Fig. 1. NU-MUSIC, MUSIC and CRB vs SNR.

from the same array and an improved estimator can be formulated by averaging the result over the M spectra as follows

$$S(\theta) = \frac{1}{\frac{1}{M} \sum_{m=1}^{M} S_m^{-1}(\theta)}$$
(25)

with

$$\hat{\theta} = \arg\max_{\theta} \mathcal{S}(\theta) \tag{26}$$

The NU-MUSIC estimator in (23) involves the eigen-decomposition of an $(M - 1) \times (M - 1)$ -dimensional covariance matrix and the averaged NU-MUSIC in (25), involves M similar decompositions. The resulting spectrum function does not involve iterative non-linear optimization and the solution reduces to a 1-D search over the range of DOAs. Note that other beamforming and high resolution estimators can be used in a similar way to MU-SIC, as the prerequisite separation between the signal and noise subspaces is provided by (21).

5. SIMULATION RESULTS

In what follows we show the global performance of the NU-MUSIC estimator and compare it to the stochastic and deterministic Cramér-Rao Bound (CRB), according to whether the data is assumed to be random or deterministic unknown, i.e., satisfying assumptions (6) or (7), respectively. In the examples the averaged NU-MUSIC is presented along with the "leave-one" NU-MUSIC resulting from the suppression of the first array element. The classical uniform MUSIC is also presented to illustrate the degradation in estimation performance due to nonuniform noise. The explicit expressions for the CRB follow from [5, 8]. Note that the CRB corresponding to the performance of the "leave-one" NU-MUSIC is evaluated with the reduced model, i.e., from covariance matrix \mathbf{R}_1 . The CRB for the augmented model corresponding to \mathbf{R} is also shown for



Fig. 2. NU-MUSIC, MUSIC and CRB vs L.

reference to assess the performance of uniform MUSIC. The figures show the CRBs for the "leave-one" NU-MUSIC and uniform MUSIC only. The assessment of the performance of the averaged NU-MUSIC against these bounds is indicative only and a proper bound for NU-MUSIC is yet to be derived. For the simulations, the true number of sources is set to P = 2 and a Uniform Linear Array (ULA) with M = 6 sensors is used. All the examples illustrate the performance in terms of the Mean Square Error (MSE) of the estimated DOAs, over 200 Monte-Carlo runs.

Figure 1 illustrates the performance with respect to the Signal to Noise Ratio (SNR). The fixed parameters are the number of snapshots L = 100 and the angles of arrival $\theta = [0^{\circ}, 45^{\circ}]^{T}$, whereas the SNR is set to vary from 0 dB to 25 dB. The noise powers are given by $\mathbf{q} = [3.3, 2.6, 5.2, 1.2, 4.1, 6.0]^{T}$, therefore the Worst Noise Power Ratio (WNPR)¹ as defined in [5] is WNPR=20. It is clear that NU-MUSIC outperforms the uniform MUSIC in nonuniform noise as the latter relies on the misleading order of the eigenvalues and a mis-modeling of the additive noise. Note that the effect of the mis-modeling is such that even at high SNR, MUSIC does not reach the CRB for the settings of the example. As expected, the averaged version of NU-MUSIC exhibits better results over the "leave-one" estimator.

Figure 2 illustrates the performance with respect to the number of snapshots L which varies from 20 to 500. The fixed parameters are SNR=10 dB and the angles of arrival $\theta = [0^{\circ}, 45^{\circ}]^{T}$. The same noise powers as for the previous example are used, thus the same WNPR applies. Similarly, MUSIC fails to retrieve the DOAs whereas NU-MUSIC takes into account the non-uniformity of the noise. The slow convergence of the algorithms to the CRB is due partly to the effect of the relatively high WNPR and MU-SIC does not converge asymptotically to the CRB for the settings of the example. The same comments as before can be made on the relative performances of the "leave-one" NU-MUSIC and the averaged NU-MUSIC.

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<sup>1</sup>In [5], WNPR = \sigma_{\text{max}}^2 / \sigma_{\text{min}}^2
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Fig. 3. NU-MUSIC, MUSIC and CRB vs $\Delta \theta$.

Figure 3 illustrates the performance with respect to the angular resolution $\Delta\theta$. The fixed parameters are SNR=10dB and the number of snapshots L = 100. The first angle of arrival is fixed at $\theta_1 = 0^\circ$ whereas the second one, θ_2 , varies from 0° to 25° . The same noise parameters as in the previous examples are used. As expected, NU-MUSIC performs better with increasing angular resolution and it outperforms uniform MUSIC. Note that the performance of MUSIC is expected to improve only if the three parameters, i.e., SNR, L and $\Delta\theta$ are increased simultaneously, otherwise the effect of nonuniform noise, through the WNPR is predominant. The same comments as before can be made on the relative performances of the "leave-one" NU-MUSIC and the averaged NU-MUSIC.

Figure 4 illustrates the performance with respect to the WNPR. The fixed parameters are SNR=10dB, the number of snapshots L = 100 and the DOAs $\theta = [0^{\circ}, 45^{\circ}]^T$. The noise parameters are chosen so that the WNPR is varied from 1 to 100. Note that the performance of NU-MUSIC degrades as the noise becomes more nonuniform. The example gives an appreciation of the limitation due to the WNPR in the previous examples. Similar comments as for the previous examples can be made on the relative performance of the uniform MUSIC, the "leave-one" NU-MUSIC and the averaged NU-MUSIC.

6. CONCLUSION

A high resolution subspace-separation-based estimator, the Non-Uniform MUSIC (NU-MUSIC), has been proposed for angle of arrival estimation in spatially nonuniform noise. The proposed estimator applies a transformation on the covariance matrix of the data, resulting from array element suppression to limit the effect of different noise powers. Simulation results have shown the accuracy of the estimation in nonuniform noise.



Fig. 4. NU-MUSIC, MUSIC and CRB vs WNPR.

7. REFERENCES

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