STOCHASTIC CRAMER-RAO BOUNDS OF DOA ESTIMATES FOR BPSK AND QPSK MODULATED SIGNALS

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ABSTRACT

This paper focuses on the stochastic Cramer-Rao bound (CRB) of direction of arrival (DOA) estimates for binary phase-shift keying (BPSK) and quaternary phase-shift keying (QPSK) modulated signals corrupted by additive circular complex Gaussian noise. Explicit expressions of the CRB for the DOA parameter alone in the case of a single signal waveform are given. Finally, these results are extended to the case of two independent BPSK distributed sources where an explicit expression of the DOA parameters alone is given for large SNR.

1. INTRODUCTION

Stochastic CRB's play an important role in DOA estimation because they serve as a benchmark for the performance of actual estimators and because the stochastic CRB can be achieved asymptotically (in the number of measurements) by the stochastic maximum likelihood (ML) method. Unfortunately, stochastic CRB's appear to be prohibitive to compute for non-Gaussian processes including discrete signal waveforms. And to the best of our knowledge, no contribution has yet dealt with stochastic CRB for non-Gaussian signal waveforms.

To cope with this difficulty, a method sometimes used is to assume that the signals are arbitrary deterministic sequences while the noise is circular complex Gaussian, so that the distribution is still Gaussian and the associated deterministic CRB is easily deduced (see e.g., [1, rel. (B.3.11)]). But the corresponding deterministic (or conditional) ML method does not achieve this deterministic CRB because the deterministic likelihood function does not meet the required regularity conditions. Consequently, this deterministic CRB is only a non-attainable lower bound on the variance of any unbiased DOA estimator. To deal with non-Gaussian processes, another solution is to suppose that the signals are Gaussian but not necessarily complex circular. In that case, the associated CRB is under rather general conditions (see e.g., [2, p. 293]) the largest CRB among the class of arbitrary distributions with given covariance matrices. This approach was used in [3] for non-circular complex signal waveforms such as discrete signals. But the associated CRB is only an upper bound on the true stochastic CRB. Faced with the drawbacks of the two aforementioned approximations, we need an explicit expression of the stochastic CRB under non-Gaussian distributions.

In this paper, we derive explicit expressions of the stochastic CRB for the DOA parameter alone in the case of BPSK and QPSK signal waveforms observed in additive circular complex Gaussian noise.

2. DATA MODEL

Consider a BPSK or QPSK modulated signal impinging on an arbitrary array of M sensors. The received signals are bandpass filtered and after down-shifting the sensor signal to baseband, the in-phase and quadrature components are paired to obtain complex signals. We assume Nyquist shaping and ideal sample timing so that the inter-symbol interference at each symbol spaced sampling instance can be ignored. In the absence of frequency offset but with possible phase offset, the signals at the output of the matched filter can be represented as:

$$\mathbf{y}_t = s_t \mathbf{a}_1 + \mathbf{n}_t \qquad t = 1, \dots, T$$

where $\mathbf{a}_1$ is the steering vector parametrized by the scalar DOA parameter θ_1. We suppose $\|\mathbf{a}_1\|^2 = M$. $s_t = \sigma_1 e^{i\phi_1}\epsilon_t$ where $(\epsilon_t)_{t=1,\dots,T}$ are independent, identically, distributed (IID) random symbols taking values ± 1 [resp. $\pm\sqrt{2}/2 \pm i\sqrt{2}/2$] with equal probabilities for BPSK [resp. QPSK] modulations, where ϕ_1 and σ_1 are considered as unknown parameters. $(\mathbf{n}_t)_{t=1,\dots,T}$ are IID M-variate zero mean complex circular Gaussian with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma_n^2\mathbf{I}_M$. Consequently $(\mathbf{y}_t)_{t=1,\dots,T}$ are IID M-dimensional random variables whose probability density function (PDF) is mixed Gaussian:

$$p(\mathbf{y}_t;\Theta) = \frac{1}{L\pi^M\sigma_n^{2M}} \sum_{l=1}^{L} e^{-\frac{\|\mathbf{y}_t - \sigma_1 e^{i\phi_1}\epsilon_l \mathbf{a}_1\|^2}{\sigma_n^2}}$$

with $L = 2$ and $\epsilon_l = \pm 1$ [resp. $L = 4$ and $\epsilon_l = \pm\sqrt{2}/2 \pm i\sqrt{2}/2$] for BPSK [resp. QPSK] modulated signals and where $\Theta \stackrel{\text{def}}{=} (\sigma_n, \sigma_1, \phi_1, \theta_1)^T$.

3. STOCHASTIC CRB FOR BPSK AND QPSK SIGNALS

Because the distribution of these models are simple mixed Gaussian, an explicit expression of the Fisher information matrix (FIM) is proved in [6] using well known properties of the Gaussian distribution.

Theorem 1 *The FIM associated with the stochastic BPSK and QPSK modulated signals are given by*

$$\mathbf{I}_{\mathrm{F}}^{\mathrm{BPSK}} = T \begin{bmatrix} \mathbf{I}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_2 \end{bmatrix}$$

with

$$\mathbf{I}_1 = \begin{pmatrix} \frac{4M}{\sigma_n^2}(1-\frac{2\sigma_1^2}{\sigma_n^2}f_1(\rho)) & \frac{4M\sigma_1}{\sigma_n^3}f_1(\rho) \\ \frac{4M\sigma_1}{\sigma_n^3}f_1(\rho) & \frac{2M}{\sigma_n^2}(1-f_1(\rho)) \end{pmatrix}$$

and

$$\mathbf{I}_2 = \begin{pmatrix} \frac{2M\sigma_1^2}{\sigma_n^2}(1-f_2(\rho)) & \frac{2\sigma_1^2|\mathbf{a}_1^H\mathbf{a}_1^{'}|}{\sigma_n^2}(1-f_2(\rho)) \\ \frac{2\sigma_1^2|\mathbf{a}_1^H\mathbf{a}_1^{'}|}{\sigma_n^2}(1-f_2(\rho)) & \frac{2\sigma_1^2\|\mathbf{a}_1^{'}\|^2}{\sigma_n^2}(1-f_2(\rho)) \end{pmatrix},$$

$$\mathbf{I}_{\mathrm{F}}^{\mathrm{QPSK}} = T \begin{bmatrix} \mathbf{I}_1^{'} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_2^{'} \end{bmatrix}$$

with

$$\mathbf{I}_1^{'} = \begin{pmatrix} \frac{4M}{\sigma_n^2}(1-\frac{2\sigma_1^2}{\sigma_n^2}f_1(\frac{\rho}{2})) & \frac{4M\sigma_1}{\sigma_n^3}f_1(\frac{\rho}{2}) \\ \frac{4M\sigma_1}{\sigma_n^3}f_1(\frac{\rho}{2}) & \frac{2M}{\sigma_n^2}(1-f_1(\frac{\rho}{2})) \end{pmatrix},$$

$$(\mathbf{I}_2^{'})_{1,1} = \frac{2M\sigma_1^2}{\sigma_n^2}\left(1-(1+\rho)f_2(\frac{\rho}{2})\right)$$

$$(\mathbf{I}_2^{'})_{1,2} = (\mathbf{I}_2^{'})_{2,1} = \frac{2\sigma_1^2|\mathbf{a}_1^H\mathbf{a}_1^{'}|}{\sigma_n^2}\left(1-(1+\rho)f_2(\frac{\rho}{2})\right)$$

$$(\mathbf{I}_2^{'})_{2,2} = \frac{2\sigma_1^2\|\mathbf{a}_1^{'}\|^2}{\sigma_n^2}\left(1-(1+\frac{\rho}{M}\frac{|\mathbf{a}_1^H\mathbf{a}_1^{'}|^2}{\|\mathbf{a}_1^{'}\|^2})f_2(\frac{\rho}{2})\right)$$

with $\rho \stackrel{\mathrm{def}}{=} \frac{M\sigma_1^2}{\sigma_n^2}$ *and* $\mathbf{a}_1^{'} \stackrel{\mathrm{def}}{=} \frac{d\mathbf{a}_1}{d\theta_1}$ *and where* f_1 *and* f_2 *are the following decreasing function of* ρ*:*

$$f_1(\rho) \stackrel{\mathrm{def}}{=} \frac{e^{-\rho}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{u^2e^{-\frac{u^2}{2}}}{\cosh(u\sqrt{2\rho})}du,$$

$$f_2(\rho) \stackrel{\mathrm{def}}{=} \frac{e^{-\rho}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{e^{-\frac{u^2}{2}}}{\cosh(u\sqrt{2\rho})}du.$$

Consequently the following explicit expressions for the CRB for the parameter DOA alone are easily derived:

$$\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1) = \frac{1}{T}\left(\frac{1}{\alpha_1}\frac{\sigma_n^2}{\sigma_1^2}\right)\left(\frac{1}{1-f_2(\rho)}\right) \quad (1)$$

$$\mathrm{CRB}_{\mathrm{QPSK}}(\theta_1) = \frac{1}{T}\left(\frac{1}{\alpha_1}\frac{\sigma_n^2}{\sigma_1^2}\right)\left(\frac{1-(1+\rho)f_2(\frac{\rho}{2})}{1-(2+\rho)f_2(\frac{\rho}{2})+(1+\rho)f_2^2(\frac{\rho}{2})}\right) \quad (2)$$

where α_1 is the purely geometrical factor $2\mathbf{a}_1^{'H}\Pi_{\mathbf{a}_1}^{\perp}\mathbf{a}_1^{'}$.

In the absence of phase offset or after correcting it (i.e. parameter ϕ_1 known), these CRB's for θ_1 become

$$\mathrm{CRB}_{\mathrm{BPSK}}^{\mathrm{CO}}(\theta_1) = \frac{1}{T}\left(\frac{1}{2\|\mathbf{a}_1^{'}\|^2}\frac{\sigma_n^2}{\sigma_1^2}\right)\left(\frac{1}{1-f_2(\rho)}\right)$$

$$\mathrm{CRB}_{\mathrm{QPSK}}^{\mathrm{CO}}(\theta_1) = \frac{1}{T}\left(\frac{1}{2\|\mathbf{a}_1^{'}\|^2}\frac{\sigma_n^2}{\sigma_1^2}\right)\left(\frac{1}{1-(1+\frac{\rho}{M}\frac{|\mathbf{a}_1^H\mathbf{a}_1^{'}|^2}{\|\mathbf{a}_1^{'}\|^2})f_2(\frac{\rho}{2})}\right).$$

Because the BPSK [resp. QPSK] modulation is non-circular [resp. circular] complex to the second-order, it makes sense to compare the stochastic CRB's (1) and (2) to the CRB's associated with respectively non-circular[1] (NCG) [3] or circular (CG) complex Gaussian distribution that can be considered as upper bounds on the true stochastic CRB's (see e.g., [2, p. 293]). More precisely, after recalling these CRB's under Gaussian distributions for the convenience of the reader

$$\mathrm{CRB}_{\mathrm{NCG}}(\theta_1) = \frac{1}{T}\left(\frac{1}{\alpha_1}\left[\frac{\sigma_n^2}{\sigma_1^2}+\frac{1}{2M}\frac{\sigma_n^4}{\sigma_1^4}\right]\right)$$

$$\mathrm{CRB}_{\mathrm{CG}}(\theta_1) = \frac{1}{T}\left(\frac{1}{\alpha_1}\left[\frac{\sigma_n^2}{\sigma_1^2}+\frac{1}{M}\frac{\sigma_n^4}{\sigma_1^4}\right]\right),$$

we have

$$\frac{\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)} = \frac{1}{(1-f_2(\rho))(1+\frac{1}{2\rho})}$$

$$\frac{\mathrm{CRB}_{\mathrm{QPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{CG}}(\theta_1)} = \frac{1-(1+\rho)f_2(\frac{\rho}{2})}{(1-(2+\rho)f_2(\frac{\rho}{2})+(1+\rho)f_2^2(\frac{\rho}{2}))(1+\frac{1}{\rho})}.$$

We note that these ratios depend on $\rho \stackrel{\mathrm{def}}{=} \frac{M\sigma_1^2}{\sigma_n^2}$ only and tend to 1 when ρ tends to ∞. However this dependence in ρ is not monotone as it is numerically shown in the next section.

[1] Because $\mathrm{E}(\epsilon_t^2) = \mathrm{E}|\epsilon_t^2|$ for the BPSK modulation, we consider the non-circular complex Gaussian distribution associated with $\mathrm{E}(\epsilon_t^2) = \mathrm{E}|\epsilon_t^2| = 1$.

4. NUMERICAL EXAMPLES

The purpose of this section is to illustrate the results of section 3 and to extend them to the case of two independent BPSK distributed sources. We consider throughout this section, one or two independent sources impinging on a uniform linear array of M sensors spaced a half-wavelength apart for which $\mathbf{a}_k = (1, e^{i\theta_k}, \dots, e^{i(M-1)\theta_k})^T$.

4.1. Single source case

The first experiment illustrates the results of section 3. Fig.1 shows the ratios $\frac{\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)}$ and $\frac{\mathrm{CRB}_{\mathrm{QPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{CG}}(\theta_1)}$ as a function of $\rho \stackrel{\text{def}}{=} \frac{M\sigma_1^2}{\sigma_n^2}$. We see from that figure that the CRB's under the non-circular [resp. circular] complex Gaussian distribution are tight upper bounds on the CRB's under the BPSK [resp. QPSK] distribution at very low and very large SNR's only.

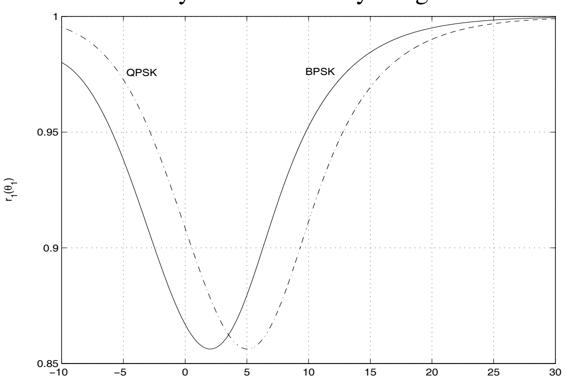

Fig.1 Ratios $r_1(\theta_1) \stackrel{\text{def}}{=} \frac{\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)}$ and $r_1(\theta_1) \stackrel{\text{def}}{=} \frac{\mathrm{CRB}_{\mathrm{QPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{CG}}(\theta_1)}$ as a function of $\rho \stackrel{\text{def}}{=} \frac{M\sigma_1^2}{\sigma_n^2}$.

4.2. Two sources case

We consider now two independent BPSK distributed sources. Because the PDF of $\mathbf{y}_t$ is a mixture of 4 Gaussian PDF's, the associated stochastic CRB appears to be prohibitive to compute. Consequently we use a numerical approximation derived from the strong low of large numbers, i.e.

$$\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1, \theta_2) = \left(\mathbf{I}_F^{-1}\right)_{(1:2,1:2)}$$

with

$$\frac{1}{T}\left(\mathbf{I}_F\right)_{k,l} = \lim_{T' \to \infty} \frac{1}{T'} \sum_{t=1}^{T'} \left(\frac{\partial \ln p(\mathbf{y}_t;\Theta)}{\partial \theta_k}\right)\left(\frac{\partial \ln p(\mathbf{y}_t;\Theta)}{\partial \theta_l}\right) \tag{3}$$

where

$$p(\mathbf{y}_t;\Theta) = \frac{1}{4\pi^M \sigma_n^{2M}} \sum_{j=1}^{4} e^{-\frac{\|\mathbf{y}_t - \mathbf{A}\mathbf{s}_j\|^2}{\sigma_n^2}}$$

with

$$\mathbf{s}_j \stackrel{\text{def}}{=} (\sigma_1\eta_{j,1}e^{i\phi_1}, \sigma_2\eta_{j,2}e^{i\phi_2})^T$$

where $(\eta_{1,1}, \eta_{1,2}) = (-1,-1)$, $(\eta_{2,1}, \eta_{2,2}) = (-1,+1)$, $(\eta_{3,1}, \eta_{3,2}) = (+1,-1)$, $(\eta_{4,1}, \eta_{4,2}) = (+1,+1)$ and where $\Theta \stackrel{\text{def}}{=} (\sigma_n, \sigma_1, \phi_1, \theta_1, \sigma_2, \phi_2, \theta_2)^T$ and $\mathbf{A} \stackrel{\text{def}}{=} (\mathbf{a}_1, \mathbf{a}_2)$. At high SNR's (more precisely for $M\frac{\sigma_1^2}{\sigma_n^2} \gg 1$ and $M\frac{\sigma_2^2}{\sigma_n^2} \gg 1$) an explicit expression of the FIM can be derived by conditioning the derivative $\frac{\partial \ln p(\mathbf{y}_t;\Theta)}{\partial \theta_k}$, $k = 1, \dots, 7$ w.r.t. the different couples $(\epsilon_{t,1}, \epsilon_{t,2}) = (\eta_{l,1}, \eta_{l,2})_{l=1,\dots,4}$ of symbols. The following expression is proved in [6]:

$$\mathbf{I}_{\mathrm{F}}^{\mathrm{BPSK}} = T \begin{bmatrix} \frac{4M}{\sigma_n^2} & \mathbf{0}^T & \mathbf{0}^T \\ \mathbf{0} & \mathbf{I}_1 & \mathbf{O} \\ \mathbf{0} & \mathbf{O} & \mathbf{I}_2 \end{bmatrix} \tag{4}$$

with $\mathbf{I}_k = \begin{bmatrix} \frac{2M\sigma_k^2}{\sigma_n^2} & \frac{2\sigma_k^2|\mathbf{a}_k^H\mathbf{a}_k^{'}|}{\sigma_n^2} \\ \frac{2\sigma_k^2|\mathbf{a}_k^H\mathbf{a}_k^{'}|}{\sigma_n^2} & \frac{2\sigma_k^2\|\mathbf{a}_k^{'}\|^2}{\sigma_n^2} \end{bmatrix}$, $k = 1, 2$.

We clearly see that the entries correponding to sources 1 and 2 are decoupled. Consequently, for large SNR's and independent sources, the CRB for the DOA of one source is independent of the parameters of the other source and

$$\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1, \theta_2) = \frac{1}{T}\begin{bmatrix} \frac{1}{\alpha_1}\frac{\sigma_n^2}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\alpha_2}\frac{\sigma_n^2}{\sigma_2^2} \end{bmatrix}.$$

Furthermore, the CRB's $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_k)$ and $\mathrm{CRB}_{\mathrm{BPSK}}^{\mathrm{CO}}(\theta_k)$ for each DOA are those of the single source case. We note that *this property is quite different from the behavior of the CRB under the Gaussian distribution and the deterministic CRB, for which the CRB for the DOA of one source depends on the DOA of the other source*. More precisely, it is proved [1, result R9] that these two CRB's tend to the same limit as all SNR's increase. For independent sources, they are given by

$$\mathrm{CRB}_{\mathrm{CG}}(\theta_1, \theta_2) = \frac{1}{T}\begin{bmatrix} \frac{1}{\beta_1}\frac{\sigma_n^2}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\beta_2}\frac{\sigma_n^2}{\sigma_2^2} \end{bmatrix}$$

with $\beta_k \stackrel{\text{def}}{=} 2\left(\|\mathbf{a}_k^{'}\|^2 - \gamma_k(\theta_1, \theta_2)\right)$, $k = 1, 2$, where

$$\gamma_k(\theta_1, \theta_2) \stackrel{\text{def}}{=} \frac{M(|\mathbf{a}_k^{'H}\mathbf{a}_k|^2 + |\mathbf{a}_k^{'H}\mathbf{a}_{3-k}|^2) - 2\Re(\mathbf{a}_k^H\mathbf{a}_k^{'}\mathbf{a}_{3-k}^H\mathbf{a}_k\mathbf{a}_k^{'H}\mathbf{a}_{3-k})}{M^2 - |\mathbf{a}_1^H\mathbf{a}_2|^2}.$$

The second experiment considers two independent and equipowered BPSK distributed sources. Fig.2 compares $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ given by (3) with the CRB under the non-circular complex Gaussian distribution. And to be fair, this comparison must be done under the same a priori that the two sources are independent. For that reason, we use the expression of the CRB obtained in [4] which can take this a priori information into account. Fig.2 exhibits the ratio $\frac{\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)}$ as a function of the DOA separation $\theta_2 - \theta_1$ for two values of the circularity phase separation $\Delta\phi \stackrel{\text{def}}{=} \phi_2 - \phi_1$. This figure shows, that contrary to the single source case, the CRB under the non-circular complex Gaussian distribution is a very loose upper bound on the CRB under the BPSK distribution except for large values of the DOA and phase separation. Consequently, *maximum likelihood (ML) solutions such as the EM approaches [5] outperform the traditional ML estimator under the Gaussian distribution specifically for small DOA and phase separation.*

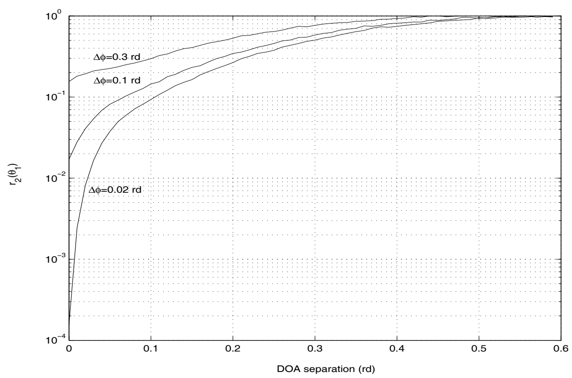

Fig.2 Ratio $r_2(\theta_1) \stackrel{\text{def}}{=} \frac{\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)}$ as a function of the DOA separation for different values of the circularity phase separation $\Delta\phi$ for $M = 6$ and $SNR = 20dB$.

Fig.3 exhibits $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ as a function of the DOA separation for two SNR's. We see that contrary to $\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$, $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ does not increase significantly when decreasing the DOA separation. This explains the behavior of $r_2(\theta_1) \stackrel{\text{def}}{=} \frac{\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)}$ for low DOA separations.

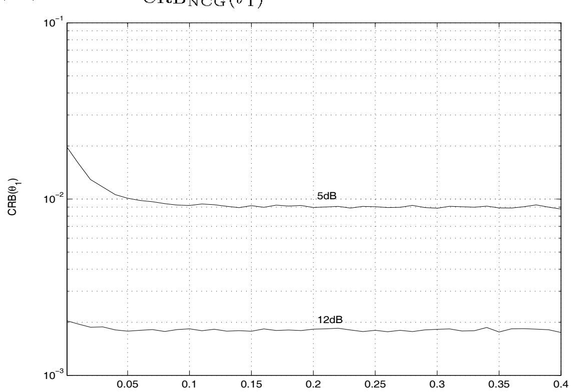

Fig.3 $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ as a function of the DOA separation for $M = 6$ and $\Delta\phi = 0.1rd$.

Finally, Fig.4 exhibits the domain of validity of the high SNR approximation (4). We see from this figure that this domain depends not only on M and SNR, but also on the DOA separation, e.g. for $M = 6$, the threshold is about 4dB for $\Delta\theta = 0.1rd$ and 8dB for $\Delta\theta = 0.05rd$. The larger the DOA separation is or the larger M is, the larger the domain of validity of the approximation is.

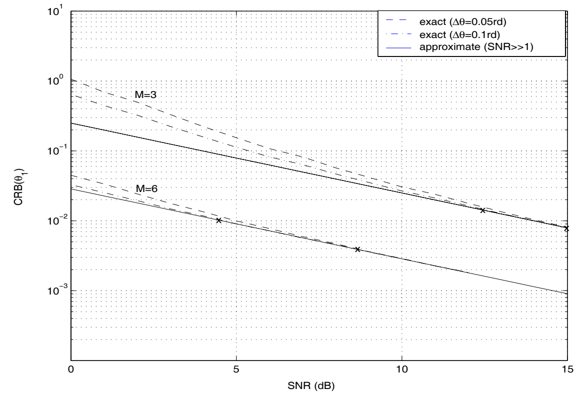

Fig.4 Approximate and exact value of $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ as a function of the SNR for different values of the DOA separation.

5. CONCLUSION

In this paper, we have proved that for a single source, the CRB's under the non-circular [resp. circular] complex Gaussian distribution are tight upper bounds on the CRB's under the BPSK [resp. QPSK] distribution at very low and very large SNR's only. For two independent BPSK sources, the CRB under the non-circular Gaussian distribution is a very loose upper bound on the CRB under the BPSK distribution. And the difference between these CRB's is more prominent for small DOA and phase separation. Furthemore for high SNR's, the CRB for the DOA of one source is independent of the parameters of the other source.

6. REFERENCES

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