Optimal FIR Approximate Inverse of Linear Periodic Filters†

Jwo-Yuh Wu and Ching-An Lin*
Department of Electrical and Control Engineering
National Chaio Tung University, Hsinchu, Taiwan
Tel: 886-35-731679; Fax: 886-35-715998; E-mail: {jywu, calin} @cc.nctu.edu.tw

Abstract *- We propose a method for constructing FIR approximate inverse for discrete-time causal FIR periodic filters in the presence of measurement noise. The objective function to be minimized is the sum of the error variance over one period. The optimization problem is formulated based on the matrix impulse response of the multi-input multi-output (MIMO) time-invariant representation of periodic filter as one that minimizes the sum of equation errors of a set of over-determined linear equations. It is shown that the problem is equivalent to a set of least squares problems and a simple closed-form solution is obtained. Numerical examples are used to illustrate the performance of the proposed FIR approximate inverse.*

I. INTRODUCTION

Periodic filters have been found useful in the areas of signal processing and communications, e.g., in subband coding [7], in speech scrambling [2], and in blind equalization [5]. The inverse, or approximate inverse, of a periodic filter is used for the recovery of scrambled signals [2] and for equalization of periodically-modulated communication channels [5]. Inverting periodic filters is discussed in [1], [3], [6] for noiseless case, and in [8], [9] when measurement noise is present.

It is well-known that associated with each N-periodic filter there is an N-input N-output time-invariant system that exhibits an input-output (I/O) relation identical to that of the filter [4], [7]. For general study of periodic systems, in particular, in the inverse filtering problem, this model is often adopted since the time-invariant nature would allow considerable simplification in analysis and design. It is known that such an MIMO filter model must satisfy certain structural constraint owing to causality of filters [4]. As a result, the design of periodic inverse filter based on the MIMO time-invariant filter model would essentially lie in finding an appropriate inverse time-invariant system subject to this constraint. Indeed, Lin and King in [3] proposed a method for finding the inverse transfer matrix in noiseless case. Recently, the approximate inverse design in the presence of noise based on the MIMO time-invariant formulation is considered in [8], [9]. For an arbitrary given periodic filter, the solution reported in [8] is in general IIR. In [9], the construction of an FIR approximate inverse for FIR periodic filters is investigated via the linear-matrix-inequality (LMI) framework.

In this paper we study the problem of constructing an FIR approximate inverse for a given FIR periodic filter when there is noise. The objective function to be minimized is the sum of variance of the approximation error over one period. There is a natural formulation of the optimization problem in terms of the matrix impulse response of the MIMO time-invariant filter model as one that minimizes the sum of equation errors of a set of over-determined linear equations. It is shown that the problem is equivalent to a set of least squares problems. Compared with the existing iterative LMI approach [9], in which essentially the same objective function is considered, our formulation thus yields simple closed-from solution: it amounts to computing a set of least squares solutions. The paper is organized as follows. Section II is the problem statement and preliminary. Section III formulates the optimization problem. Section IV derives the solution. Section V is the simulation results. Finally, Section VI is the conclusion.

† Research sponsored by National Science Council under grant NSC-91-2219-E-009-049.
* Corresponding author.

Notation List: We denote by $\Re^{l\times m}$ the set of all $l\times m$ real matrices. The notations $0_{l\times m}$ and I_l respectively stand for the $l\times m$ zero matrix and the $l\times l$ identity matrix. Denote by δ the $N\times N$ matrix unit-impulse sequence, i.e., $\delta_n = 0_{N\times N}$ for all $n>0$ and $\delta_0 = I_N$. Let z^{-k} be the k-step delay operator such that, for any sequence s, $(z^{-k}s)_n = s_{n-k}$. Let $H^{l\times m}$ be the space of all causal sequences of matrices $X=\{X_n \in \Re^{l\times m}, n\ge 0\}$. Given a positive integer K, define the subspace $H_K^{l\times m} := \{X\in H^{l\times m} : X_n = 0_{l\times m} \text{ for all } n\ge K\}$. The norm of $X\in H_K^{l\times m}$ is defined by $\|X\|^2 := \sum_{n=0}^{K-1}\|X_n\|_F^2$, where $\|\cdot\|_F$ is the Frobenius norm. For $X\in H_{K_1}^{l\times m}$ and $Y\in H_{K_2}^{l\times m}$, let the augmented sequence $[X \quad Y]\in H_{K_3}^{l\times 2m}$ be such that $[X \quad Y]_n = [X_n \quad Y_n]$ and $K_3 = \max\{K_1, K_2\}$.

II. PROBLEM STATEMENT AND PRELIMINARY

A. Problem Statement

Consider the discrete-time causal FIR N-periodic filter with input u and output z described by

$$z_n = \sum_{k=0}^{M} g_{n,k}u_{n-k}, \quad n\ge 0, \tag{2.1}$$

where u_n and z_n are respectively the input and output at time n, and the filter coefficient satisfies

$$g_{n,k} = g_{n+N,k}, \quad \forall n\ge 0, \quad 0\le k\le M. \tag{2.2}$$

Let r be the observed signal, which is the sum of the filter output z and a measurement noise v, i.e., $r = z+v$, and $\hat{u}$ be the d-step delay of the input u to filter (2.1), that is,

$$\hat{u}_n = \begin{cases} u_{n-d}, & n\ge d, \\ 0, & 0\le n< d. \end{cases} \tag{2.3}$$

An *FIR approximate inverse* of filter (2.1) is a causal N-periodic filter with input r and output y described by

$$y_n = \sum_{k=0}^{M_1} f_{n,k} r_{n-k} , \tag{2.4}$$

where, for each $0 \le k \le M_1$,

$$f_{n,k} = f_{n+N,k} , \quad \forall n \ge 0 , \tag{2.5}$$

such that the output y is close to $\hat{u}$, i.e., the error signal

$$e = \hat{u} - y \tag{2.6}$$

is small for the input signal u of interest.

The following assumptions are made in the sequel.

1) The input $u = \{u_n \in \Re, n \ge 0\}$ to filter (2.1) is a white sequence with zero-mean and unit variance.

2) The noise $v = \{v_n \in \Re, n \ge 0\}$ is a white sequence with zero-mean and variance σ_v^2, and is uncorrelated with u.

In this paper we propose a method for constructing an FIR approximate inverse of the form (2.4), with which the sum of error variance over one period is minimized.

B. Matrix Impulse Response of FIR Periodic Filters

Consider again the filter (2.1). Define the block input $\bar{u}$ and output $\bar{z}$ as

$$\bar{u}_n := [u_{nN} \ \ u_{nN+1} \ \cdots \ u_{nN+N-1}]^T \in \Re^N , \quad n \ge 0 , \tag{2.7}$$

and

$$\bar{z}_n := [z_{nN} \ \ z_{nN+1} \ \cdots \ z_{nN+N-1}]^T \in \Re^N , \quad n \ge 0 . \tag{2.8}$$

It is well known that associated with filter (2.1) there is an N-input N-output time-invariant system, with input $\bar{u}$ and output $\bar{z}$, that exhibits an input-output relation identical to that of filter (2.1) [4]. Based on the periodic difference equation (2.1), there is a simple procedure for obtaining the matrix impulse response of the associated MIMO time-invariant system. This can can be seen as follows.

Write the ith component of $\bar{z}_n$ as (see (2.1))

$$z_{nN+i} = \sum_{k=0}^{M} g_{nN+i,k} u_{nN+i-k} = \sum_{k=0}^{M} g_{i,k} u_{nN+i-k} , \quad 0 \le i \le N-1 , \tag{2.9}$$

where the second equality follows from (2.2). Let $m := M \text{ modulo } N$, thus $0 \le m \le N-1$, and choose

$$L = \lceil M/N \rceil + 1 , \tag{2.10}$$

where $\lceil M/N \rceil$ is the smallest integer which is greater than or equal to M/N. Collecting z_{nN+i} in (2.9), $0 \le i \le N-1$, into a vector and by rearrangement, we can express $\bar{z}_n$ in (2.8) as the following product form:

$$\bar{z}_n = GU , \tag{2.11}$$

where $G \in \Re^{N \times LN}$ is the filter coefficient matrix whose ith row, $0 \le i \le N-1$, is

$$\begin{cases} [0_{1\times i} \quad g_{i,M} \quad \cdots \quad g_{i,0} \quad 0_{1\times(N-1-i)}], & \text{if } m=0, \\ [0_{1\times(N-m+i)} \quad g_{i,M} \quad \cdots \quad g_{i,0} \quad 0_{1\times(N-1-i)}], & \text{if } 1 \le m \le N-1; \end{cases} \tag{2.12}$$

$U \in \Re^{LN}$ is the vector containing the input samples having contribution to $\bar{z}_n$ and is given as

$$U = \begin{cases} [u_{nN-M} \cdots u_{uN-1} \, u_{nN} \cdots u_{nN+N-1}]^T, & \text{if } m=0, \\ [u_{nN-M-(N-m)} \cdots u_{nN-M} \cdots u_{nN} \cdots u_{nN+N-1}]^T, & \text{if } 1 \le m \le N-1. \end{cases} \tag{2.13}$$

We note that the leading zero entries in the row vectors given in (2.12) result from the fact that filter (2.1) is FIR with order M; the tailing zero entries are due to the causality of filter (2.1). In terms of block input $\bar{u}_n$ in (2.7), it can be checked that the input sample vector U in (2.13), for $0 \le m \le N-1$, is equal to

$$U = [\bar{u}_{n-(L-1)}^T \cdots \ \bar{u}_{n-1}^T \ \bar{u}_n^T]^T . \tag{2.14}$$

Partition the matrix G defined in (2.12) as

$$G = [\mathsf{G}_{L-1} \quad \cdots \quad \mathsf{G}_0] , \tag{2.15}$$

where $\mathsf{G}_l \in \Re^{N \times N}$, $0 \le l \le L-1$. With (2.14) and (2.15), the product expression of $\bar{z}_n$ in (2.11) can be written in the convolutional form as $\bar{z}_n = \sum_{l=0}^{L-1} \mathsf{G}_l \bar{u}_{n-l}$. Hence the MIMO time-invariant system associated with filter (2.1) is described by $\bar{z} = \mathsf{G} * \bar{u}$, where $*$ denotes the convolution and the matrix impulse response G which characterizes the system is

$$\mathsf{G} = \sum_{l=0}^{L-1} \mathsf{G}_l z^{-l} \delta \ \in H_L^{N \times N} . \tag{2.16}$$

As a result, an Mth order FIR N-periodic filter of the form (2.1) is represented by a $\mathsf{G} \in H_L^{N \times N}$ as in (2.16), where L is given in (2.10) and $\mathsf{G}_l \in \Re^{N \times N}$ is defined through the filter coefficient matrix G as in (2.15). Conversely, for a given $\mathsf{G} \in H_L^{N \times N}$, if we form the associated G matrix according to (2.15) and if its rows are of the form (2.12), then G can be implemented as a single-input single-output (SISO) FIR N-periodic filter of the form (2.1). In particular, the $M+1$ nonzero entries in the ith row of G, $0 \le i \le N-1$, yield the filter coefficients $g_{i,k}$ for $0 \le k \le M$. In the sequel, we will simply call G the *matrix impulse response* of filter (2.1). The matrix impulse response of the d-step delay is given as follows.

Proposition 2.1 [3]: The matrix impulse response associated with the d-step delay, when regarded as an N-periodic system, is

$$\mathsf{D} = \sum_{n=q}^{q+1} \mathsf{D}_n z^{-n} \delta \in H_{q+2}^{N \times N} , \tag{2.17}$$

where, $d = p + qN$, p and q are nonnegative integers with $0 \le p \le N-1$,

$$\mathsf{D}_q = \begin{bmatrix} 0 & 0 \\ I_{N-p} & 0 \end{bmatrix} \in \Re^{N \times N} \quad \text{and} \quad \mathsf{D}_{q+1} = \begin{bmatrix} 0 & I_p \\ 0 & 0 \end{bmatrix} \in \Re^{N \times N} .$$

□

III. OPTIMIZATION PROBLEM

The optimality criterion is based on the fact that the error variance $E|e_n|^2$ is N-periodic for n large enough. Specifically, let the error signal e be as defined in (2.6). Then we have

$$E|e_n|^2 = E|e_{n+N}|^2 , \quad \forall \ n \ge M + M_1 . \tag{3.1}$$

Equation (3.1) can be readily obtained by assumptions *1)* and *2)*, and using (2.2) and (2.5). From (3.1), the sum of $E|e_n|^2$ over an arbitrary block of N samples within the time range $n \geq M + M_1$ is thus a constant independent of which block is chosen for summation. This suggests the following objective function

$$J := \sum_{n=M+M_1}^{M+M_1+N-1} E|e_n|^2 \ . \tag{3.2}$$

If J is small, then $E|e_n|^2$ is small for each $M + M_1 \leq n \leq M + M_1 + N - 1$. From (3.1), it follows that $E|e_n|^2$ is small for *all* $n \geq M + M_1$. The block invariant property of J will also enable us to analyze the optimization problem by using the MIMO time-invariant representation of periodic filters. Moreover, since the objective function J is quadratic in nature, the optimization problem, potentially, could be relatively easy to solve. Hence we propose to find an approximate inverse by minimizing the objective function J.

To express the objective function J in (3.2) in terms of the MIMO time-invariant filter models, let the matrix impulse response of filter (2.4) be $F = \sum_{l=0}^{L_1-1} F_l z^{-l}\delta \in H_{L_1}^{N\times N}$, where $L_1 = \lceil M_1 / N \rceil + 1$, and $F_l \in \Re^{N\times N}$, $0 \leq l \leq L_1 - 1$, contain the unknown filter coefficients $f_{i,k}$ for $0 \leq i \leq N-1$ and $0 \leq k \leq M_1$. Define $L_2 := L + L_1 - 1$ and thus, by definition of convolution, L_2 is the duration of the sequence $F * G$, where G is the matrix impulse response of filter (2.1). Assume that L_1 is chosen so that $L_2 \geq q + 2$, i.e., the signal duration of $F * G$ is no less than that of the delay D (see (2.17)). Then it can be shown that the objective function J defined in (3.2) can be expressed as

$$J = \| D - F * G \|^2 + \sigma_v^2 \| F \|^2 . \tag{3.3}$$

Hence the optimization problem, in terms of matrix impulse responses, is to find an $F \in H_{L_1}^{N\times N}$, which can be realized as an SISO FIR N-periodic filter of the form (2.4), so as to minimize J defined in (3.3).

Remark: The assumption $L_2 \geq q + 2$ is necessary, since otherwise the quantity $\| D - F * G \|^2$ in general can not be made small. This is because, if $L_2 < q + 2$, then $(F * G)_{q+1} = 0_{N\times N}$ and thus can not be kept close to D_{q+1} by choosing any $F \in H_{L_1}^{N\times N}$. □

IV. OPTIMAL SOLUTION

In this section we derive the optimal solution. Section 4.1 formulates the optimization problem in terms of linear equations. Section 4.2 then derives the solution.

4.1 Linear Equations Formulations

Associated the matrix impulse responses G and D, define the respective augmented sequences

$$\hat{G} := [G \quad -\sigma_v \delta] \in H_L^{N\times 2N} , \tag{4.1}$$

$$\hat{D} := [D \quad 0_{N\times N}] \in H_{L_2}^{N\times 2N} . \tag{4.2}$$

Then, with (3.3) and by definition of $\| \cdot \|$, it can be checked that

$$J = \| \hat{D} - F * \hat{G} \|^2 \ . \tag{4.3}$$

Based on (4.3), the objective function J in (3.3) can be directly expressed in terms of the N rows of the filter coefficient matrix associated with filter (2.4); this will allow a problem formulation in terms of a set of linear equations. Indeed, from (4.3) and by definition of Frobenius norm, it follows that

$$J = \sum_{n=0}^{L_2-1} \| \hat{D}_n - (F * \hat{G})_n \|_F^2 = \| D^T - [(F * \hat{G})_0 \cdots (F * \hat{G})_{L_2-1}] \|_F^2 , \tag{4.4}$$

where

$$D^T := [\hat{D}_0 \cdots \hat{D}_{L_2-1}] \in \Re^{N\times 2L_2 N} . \tag{4.5}$$

Since $(F * \hat{G})_n = \sum_{l=0}^{L_1-1} F_l \hat{G}_{n-l}$, with some manipulations we have

$$[(F * \hat{G})_0 \cdots (F * \hat{G})_{L_2-1}] = X^T A^T , \tag{4.6}$$

where

$$X^T := [F_{L_1-1} \quad \cdots \quad F_0] \in \Re^{N\times L_1 N} ; \tag{4.7}$$

$A^T \in \Re^{L_1 N\times 2L_2 N}$ is the $N \times 2N$ - *block* Hankel matrix with $[0_{2N\times N} \cdots 0_{2N\times N} \ \hat{G}_0^T]^T \in \Re^{L_1 N\times 2N}$ as the first block column and $[0_{N\times 2N} \cdots 0_{N\times 2N} \ \hat{G}_0 \cdots \hat{G}_{L-1}] \in \Re^{N\times 2L_2 N}$ as the first block row. We should note that, since the sequence F is the matrix impulse response of filter (2.4), for each $0 \leq i \leq N-1$, the ith row X_i^T of X^T in (4.7) is thus of the form (2.12), viz., for $m_1 = M_1$ modulo N,

$$X_i^T = \begin{cases} [0_{1\times i} \quad f_{i,M_1} \quad \cdots \quad f_{i,0} \quad 0_{1\times(N-1-i)}], & \text{if } m_1 = 0, \\ [0_{1\times(N-m_1+i)} \quad f_{i,M_1} \quad \cdots \quad f_{i,0} \quad 0_{1\times(N-1-i)}], & \text{if } 1 \leq m_1 \leq N-1. \end{cases} \tag{4.8}$$

With (4.4) and (4.6), we immediately have $J = \| D^T - X^T A^T \|_F^2 = \| D - AX \|_F^2$, where the last equality follows since the Frobenius norms of a matrix and its transpose are the same. Again by definition of Frobenius norm, it can be further checked that

$$J = \sum_{i=0}^{N-1} \| AX_i - D_i \|_F^2 , \tag{4.9}$$

where $D_i \in \Re^{2L_2 N}$ and $X_i \in \Re^{L_1 N}$ are respectively the ith columns of the matrices D and X (in (4.5) and (4.7)). With (4.9), the objective function J is thus minimized if, for *each* $0 \leq i \leq N-1$, we can find an X_i of the form (4.8), or equivalently, $M_1 + 1$ unknown filter coefficients $f_{i,k}$ ($0 \leq k \leq M_1$) since the remaining entries in X_i are zero, that minimize $\| AX_i - D_i \|_F^2$. This is done in the next subsection.

4.2 Optimal Solution

Since X_i defined in (4.8) has only $M_1 + 1$ non-zero entries, the product AX_i simplifies to a linear combination of $M_1 + 1$ columns of A. As a result, each group of equations $AX_i \approx D_i$ contains a set of $2L_2 N$ scalar equations in $M_1 + 1$ unknowns. Based on this observation, the optimization problem can be reduced to a set of N least squares problems, whose solutions are very easy to compute. To be specific, for $0 \leq i \leq N-1$, write

$Y_i := [f_{i,M_1} \quad \cdots \quad f_{i,0}]^T \in \Re^{M_1+1}$ be the ith filter coefficient vector. For $0 \le i \le N-1$, let $A_i \in \Re^{2L_2N\times(M_1+1)}$ be the matrix obtained from A by deleting its first i (or first $N-m_1+i$, if $m_1 \ne 0$) columns and last $N-1-i$ columns. For any X_i of the form (4.8), it follows immediately that $AX_i = A_iY_i$, for each $0 \le i \le N-1$. Since Y_i is arbitrary, the optimization problem is thus equivalent to $\min_{Y_i} \| A_iY_i - D_i \|_F^2$, $0 \le i \le N-1$. Assume that each A_i is of full column rank (the condition holds in general owing to the block Hankel structure of the matrix A). The optimal $\widetilde{Y}_i$ is then computed as

$$\widetilde{Y}_i = \left(A_i^T A_i\right)^{-1} A_i^T D_i, \quad 0 \le i \le N-1. \qquad (4.10)$$

Compared with [9], in which the resultant solution is obtained via the LMI method, the proposed approach leads to a relatively simple closed-form solution as in (4.10).

V. SIMULATION RESULTS

In this section, numerical examples are used to illustrate the performance of the proposed FIR approximate inverse. For each conducted Monte-Carlo realization, the number of input samples to filter (2.1) is 100; the number of independent trials is $I = 1000$. In the first simulation we illustrate the effect of M_1, the order of FIR approximate inverse, on performance. Consider the following 2-periodic filter

$$\begin{cases} g_{n,0} = 1.2,\ g_{n,1} = 2,\quad g_{n,2} = -0.1555,\ g_{n,3} = 0.3318, & \text{for even } n; \\ g_{n,0} = 0.8,\ g_{n,1} = -2.4,\ g_{n,2} = -0.1037,\ g_{n,3} = 0.4976, & \text{for odd } n. \end{cases}$$

We fix reconstruction delay at $d = 6$ and consider the two cases SNR=0 dB and 10 dB. For each $3 \le M_1 \le 20$, an approximate inverse is designed using (4.10). Figure 1 shows the computed time average objective function versus M_1, with respect to the two SNR levels. The respective theoretical values of the objective function J computed using (4.9) are also shown. The result shows that the experimental values are almost identical to the theoretical values. We can also see that the performance is improved by increasing M_1. In particular, as M_1 increases, the approximation error tends to decrease toward a lower bound, whose value depends on SNR. In the second simulation, we illustrate the effect of reconstruction delay d on performance. We consider the 2-periodic filter used in the previous simulation and fix SNR at 10 dB. For each $0 \le d \le 11$, an approximate inverse with order M_1 large enough is first designed so as to compute the associated lower bound of J with respect to this particular SNR. We find that the choice $M_1 = d + 6$ suffices to yield J fairly close to the computed lower bound associated with each delay. With such choice of M_1, Figure 2 shows the computed time average of J, together with the theoretical J, for $0 \le d \le 11$. As we can see, the performance is improved as d increases. It seems that there is a best achievable performance (-12.3 dB in our case) under the fixed SNR, no matter how large d is used.

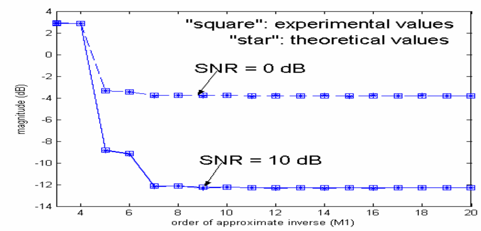

Figure 1 Approximation error versus order, delay d=6.

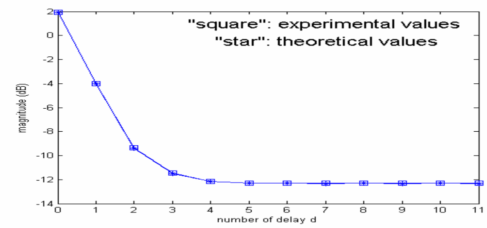

Figure 2 Approximation error versus delay, SNR=10 dB.

VI. CONCLUSION

We propose a method for constructing an FIR approximate inverse for a given FIR periodic filter in the presence of measurement noise. The adopted optimality criterion, which minimizes the sum of error variances over one period, allows us to formulate the problem in time domain in terms of the matrix impulse responses of MIMO time-invariant representation of periodic filters. There is a simple procedure for obtaining the matrix impulse response directly from the coefficients of filters. The resultant optimization problem is essentially solving a set of least squares problems. The computations required are computing N least squares solutions but does not involve numerical optimization as is required in the existing LMI approach [9].

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