LOWER BOUNDS ON THE VARIANCE OF DETERMINISTIC SIGNAL PARAMETER ESTIMATORS USING BAYESIAN INFERENCE

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ABSTRACT

Bayesian estimators have applied to both random and deterministic parameters. In the Bayesian estimation of deterministic parameters, the randomness is introduced only through the observations and the prior distributions are adopted to impose certain constraints. In such cases, neither the well known Cramer-Rao lower bound (CRLB) or the posterior CRLB can be used reasonably as the performance lower bounds. In this paper, the theory of CRLB is extended under the Bayesian framework to provide the lower bounds for both unbiased and biased Bayesian estimators of deterministic parameters. An example is provided to show the effectiveness of the proposed lower bound over other popular lower bounds.

1. INTRODUCTION

Cramer-Rao lower bound (CRLB) [1, 2] is a widely used on the variances of unbiased estimators. It was developed under a frequentist set up where the unknowns are considered deterministic and inferences are drawn from the likelihood distributions. When the unknowns are random, the Bayesian methodology is usually adopted and the counterpart of CRLB herein is known as the posterior CRLB (PCRLB), which is the lower bound on the posterior variance of estimators [1, 3].

Theoretically, the Bayesian methods are only applicable to problems whose unknown parameters are random. However, in engineering applications, they have been employed for the estimation of deterministic parameters as well [4, 5]. There, the deterministic unknowns are treated as if they were random and the prior distributions adopted can be interpreted as constraints. By no means, the priors indicate the randomness of the unknowns and the randomness is only introduced by the observations. Consequently, the performance of these Bayesian estimators for deterministic parameters should be evaluated through the variance or the mean square error (MSE) with respect to the likelihood function rather than the posterior variance. In fact, in such cases, it is a common practice to use MSEs to compare Bayesian estimators with the maximum likelihood estimators (MLE)[4, 5]. However, for Bayesian estimators, since inferences are drawn from the posterior distributions, CRLB can not be used as a low bound any more. To assess the performance of Bayesian estimators for the deterministic parameters, and especially, to provide the comparison with MLE, it is of both practical and theoretical importance to formulate a lower bound on the variance or MSE. To the authors' best knowledge, no such bounds have been reported in the literature and it is the objective of this paper to establish such bounds.

We first briefly introduce the basic concept of Bayesian inference as well as CRLB and PCRLB. We then derive the desired lower bounds for both unbiased and biased estimators for the scalar and vector parameters. In addition, we also discuss the case when nuisance parameters exist. Finally, an example is presented to illustrate effectiveness of the obtained bounds.

2. THE PARAMETRIC INFERENCE AND LOWER BOUNDS OF ESTIMATORS

Suppose that a vector of noisy observation $\mathbf{y}$ is generated from some model parameterized by a vector of the unknowns $\boldsymbol{\theta}$. The objective of the parametric inference is to provide an accurate estimate of the unknowns $\boldsymbol{\theta}$. In non-Bayesian approaches such as MLEs, the unknowns $\boldsymbol{\theta}$ are considered as deterministic, and inferences of$\boldsymbol{\theta}$ are drawn from the likelihood distribution $p(\mathbf{y}|\boldsymbol{\theta})$. The performances of unbiased estimators are usually evaluated by comparing the variances or the covariance matrix of these estimators with lower bounds like CRLB. To introduce CRLB, suppose that the covariance matrix of the estimate $\hat{\boldsymbol{\theta}}$ is defined by

$$\boldsymbol{C}(\hat{\boldsymbol{\theta}}) = E_{\mathbf{y}}[(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathsf{T}}] \tag{1}$$

where $E_{\mathbf{y}}[\cdot]$ denotes the expectation with respect to the likelihood function $p(\mathbf{y}|\boldsymbol{\theta})$. Then, CRLB indicates that if $p(\mathbf{y}|\boldsymbol{\theta})$ satisfies the Wolfowitz's regularity condition [2], the covariance matrix $\mathbf{C}(\hat{\boldsymbol{\theta}})$ satisfies

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} \geq \mathbf{I}^{-1}(\boldsymbol{\theta}) \tag{2}$$

where $\mathbf{I}(\boldsymbol{\theta})$ denotes the Fisher information matrix which is given by

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = -E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j}]. \tag{3}$$

The estimators which achieve CRLB are usually said to be *efficient*.

On the other hand, in Bayesian approaches, the unknown parameters $\boldsymbol{\theta}$ are considered as random and inferences are drawn from the posterior distribution $p(\boldsymbol{\theta}|\mathbf{y})$ which is proportional to the product of $p(\mathbf{y}|\boldsymbol{\theta})$ and the prior distribution $p(\boldsymbol{\theta})$. The minimum mean square error (MMSE) estimator

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and maximum *a posterior* are the most popular Bayesian estimators. The Bayesian analogy of the CRLB is called posterior CRLB which is the lower bound on the Bayesian MSE and has the form

$$E_{\mathbf{y},\boldsymbol{\theta}}[(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathsf{T}}] \geq \mathbf{I}_T^{-1} \tag{4}$$

where $\mathbf{I}_T$ is an information matrix with ij-th element defined as

$$\{\mathbf{I}_T\}_{ij} = -E_{\mathbf{y},\boldsymbol{\theta}}[\frac{\partial^2 \ln p(\mathbf{y}, \boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}]. \tag{5}$$

Notice that the expectation here is with respect to both $p(\mathbf{y}|\boldsymbol{\theta})$ and $p(\boldsymbol{\theta})$, hence PCRLB is independent of $\boldsymbol{\theta}$.

3. THE LOWER BOUNDS OF BAYESIAN ESTIMATIOR FOR DETERMINISTIC PARAMETERS

In practice, Bayesian approaches are often used to estimate deterministic parameters. Although in such cases, the parameters are treated as if they were random, the randomness of the problem still only comes from the data like in any deterministic scenario. Hence, the variance rather than the Bayesian MSE of the estimators would be the characteristics for evaluating the performance of the Bayesian estimators. However, since CRLB does not consider the contribution from the prior distribution $p(\boldsymbol{\theta})$, it cannot be used as the lower bound for the Bayesian estimators. In what follows, the theory of CRLB is extended, under the Bayesian framework, to provide the lower bounds of the Bayesian estimators for deterministic parameters.

3.1. The lower bounds for the scalar parameter

Let θ and $\hat{\theta}_B$ represent the unknown scalar parameter and its Bayesian estimate. Since θ is deterministic, the bounds desired are on $E_{\mathbf{y}}[(\hat{\theta}_B - \theta)^2]$ which defines the variance of $\hat{\theta}_B$ when $\hat{\theta}$ is unbiased, or MSE when $\hat{\theta}$ is biased. Before deriving the lower bounds, we would like to first introduce the following proposition.

Proposition 1

$$E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta}]^2 = -E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2. \tag{6}$$

Proof: Consider a trivial equality

$$\int p(\mathbf{y}, \theta) d\mathbf{y} = p(\theta) \tag{7}$$

Differentiating both sides of (7) over θ, and considering the transformations of $\frac{\partial p(\mathbf{y},\theta)}{\partial \theta} = \frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta} p(\mathbf{y}, \theta)$ and $\frac{\partial p(\theta)}{\partial \theta} = \frac{\partial \ln p(\theta)}{\partial \theta} p(\theta)$, we have

$$\int \frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta} p(\mathbf{y}|\theta) d\mathbf{y} = \frac{\partial \ln p(\theta)}{\partial \theta}. \tag{8}$$

Now, differentiate (8) over θ again , we have

$$\int \frac{\partial^2 \ln p(\mathbf{y}, \theta)}{\partial \theta^2} p(\mathbf{y}|\theta) d\mathbf{y} + \int \frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta} \frac{\partial \ln p(\mathbf{y}|\theta)}{\partial \theta} p(\mathbf{y}|\theta) d\mathbf{y} = \frac{\partial^2 \ln p(\theta)}{\partial \theta^2} \tag{9}$$

which may be further written as

$$\int \frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2} p(\mathbf{y}|\theta) d\mathbf{y} + \int \frac{\partial^2 \ln p(\theta)}{\partial \theta^2} p(\mathbf{y}|\theta) d\mathbf{y} + \int (\frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta})^2 p(\mathbf{y}|\theta) d\mathbf{y} - \frac{\partial \ln p(\theta)}{\partial \theta} \int \frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta} p(\mathbf{y}|\theta) d\mathbf{y} = \frac{\partial^2 \ln p(\theta)}{\partial \theta^2}. \tag{10}$$

In review of (8), we then have

$$E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta}]^2 - (\frac{\partial \ln p(\theta)}{\partial \theta})^2 = 0 \tag{11}$$

or

$$E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta}]^2 = -E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2. \tag{12}$$

◇

Now let us first consider an unbiased Bayesian estimator $\hat{\theta}$. The following theorem provides a lower bound on the variance of $\hat{\theta}$.

Theorem 1 (Lower Bound of the Unbiased Bayesian Estimator for the Deterministic Scalar Parameter) *It is assumed that $p(\mathbf{y}|\theta)$ satisfies the regularity condition*

$$E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y}|\theta)}{\partial \theta}] = 0 \quad \textit{for all } \theta. \tag{13}$$

Then, the variance of any unbiased Bayesian estimator must satisfy

$$var(\hat{\theta}) \geq \left(-E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2\right)^{-1}. \tag{14}$$

Furthermore, an unbiased Bayesian estimator may be found that attains the bound for all θ if and only if

$$\frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta} = k(\theta)(g(\theta) - \theta) \tag{15}$$

for some functions $k(\cdot)$ and $g(\cdot)$. That estimator is $\hat{\theta} = g(\theta)$ and the minimum variance is $1/k(\theta)$.

Proof: Since the estimator is unbiased, we have

$$E_{\mathbf{y}}(\hat{\theta} - \theta) = 0 \tag{16}$$

or

$$\int (\hat{\theta} - \theta)) p(\mathbf{y}|\theta) d\mathbf{y} = 0 \tag{17}$$

After multiplying the prior $p(\theta)$ to the both side of (17) and differentiating over θ, we have

$$-\int p(\mathbf{y}, \theta) d\mathbf{y} + \int (\hat{\theta} - \theta) \frac{\partial p(\mathbf{y}, \theta)}{\partial \theta} d\mathbf{y} = 0. \tag{18}$$

Since $\int p(\mathbf{y}|\theta) dy = 1$, (18) can then be expressed as

$$\int (\hat{\theta} - \theta) \frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta} p(\mathbf{y}, \theta) d\mathbf{y} = p(\theta). \tag{19}$$

Applying the Schwarz inequality on (19), we obtain

$$\int (\hat{\theta}-\theta)^2 p(\mathbf{y},\theta) d\mathbf{y} \int (\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta})^2 p(\mathbf{y},\theta) d\mathbf{y} \geq p^2(\theta)$$

or

$$\begin{aligned} var(\hat{\theta}) &\geq \left(E_{\mathbf{y}}[(\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta}]^2 \right)^{-1} \\ &= \left(-E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2 \right)^{-1} . \end{aligned} \quad (20)$$

The last equality is the direct result of Proposition 1. Note that the condition for the equality in (20) is

$$\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta} = k(\theta)(\hat{\theta}-\theta) \quad (21)$$

for some function $k(\cdot)$, which is the condition for $\hat{\theta}$ to attain the lower bounds. $\diamondsuit$

Comparing the lower bound (14) with the CRLB, we see that they differ by the second term $(\frac{\partial \ln p(\theta)}{\partial \theta})^2$ in the denominator of (14) and since this term is nonnegative, we thus have the following corollary.

Corollary 1 *Let $\hat{\theta}_B$ denote an unbiased Bayesian estimator who attains the lower bound (14) and $\hat{\theta}_E$ denote an efficient estimator who attains the CRLB. Then*

$$var(\hat{\theta}_B) \leq var(\hat{\theta}_E). \quad (22)$$

The above corollary indicates that as long as we can obtain an unbiased Bayesian estimator which attains (14), it will have smaller variance than an efficient estimator. However, in practice the Bayesian estimators are often biased which leads us to provide the lower bound for the biased estimators.

Theorem 2 (Lower Bound of the Biased Bayesian Estimator for the Deterministic Scalar Parameter) *It is assumed that $p(\mathbf{y}|\theta)$ satisfies the regularity condition (13). Then, the MSE of any biased Bayesian estimator must satisfy*

$$mse(\hat{\theta_B}) \geq \frac{(1+\frac{\partial \ln(b(\theta)p(\theta))}{\partial \theta} b(\theta))^2}{-E_{\mathbf{y}}[\frac{\partial^2 \ln p(\mathbf{y}|\theta)}{\partial \theta^2}] + (\frac{\partial \ln p(\theta)}{\partial \theta})^2}. \quad (23)$$

where $b(\theta)$ is the bias of the estimator $\hat{\theta}$.

Apparently, this is a bound on MSE of the estimator.

3.2. The lower bounds for the vector parameters

In this section, we extend the above discussion to provide the bounds of the Bayesian estimators for the deterministic vector parameters. Due to lack of space, the theorems that follow are presented without the proof .

Theorem 3 (Lower Bound of Uniased Bayesian Estimators for the Deterministic Vector Parameter) *It is assumed that $p(\mathbf{y}|\boldsymbol{\theta})$ satisfies the regularity condition*

$$E_{\mathbf{y}}[\frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \theta}] = 0 \quad \textit{for all } \boldsymbol{\theta}. \quad (24)$$

Then, the covariance matrix of any unbiased Bayesian estimator $\hat{\boldsymbol{\theta}}_B$ must satisfy

$$\begin{aligned} \mathbf{C}(\hat{\boldsymbol{\theta}}_B) &\geq \mathbf{I}_B^{-1}(\boldsymbol{\theta}) \\ &= (\mathbf{I}(\boldsymbol{\theta}) + \frac{\partial \ln p(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \frac{\partial \ln p(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}^{\mathsf{T}})^{-1} \end{aligned} \quad (25)$$

where $\mathbf{I}$ is the Fisher information matrix. Furthermore, the unbiased Bayesian estimators may be found that attain the bound for all $\boldsymbol{\theta}$ if and only if

$$\frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{I}_B(\boldsymbol{\theta})(\mathbf{g}(\boldsymbol{\theta}) - \boldsymbol{\theta}) \quad (26)$$

for some function $\mathbf{g}(\cdot)$. That estimator is $\hat{\boldsymbol{\theta}}_B = \mathbf{g}(\theta)$ and its covariance matrix is $\mathbf{I}_B^{-1}(\boldsymbol{\theta})$.

Note that the positive semidefinite of $\mathbf{I}_B(\boldsymbol{\theta})$, a direct implication from the Theorem 3 by the lower bound on the variance of each component of $\hat{\boldsymbol{\theta}}_B$ is

$$var([\hat{\boldsymbol{\theta}}_B]_i) \geq [\mathbf{I}_B^{-1}]_{ii} \qquad i = 1, 2, \cdots, N \quad (27)$$

Theorem 4 (Lower Bound of Biased Bayesian Estimators for the Deterministic Vector Parameter) *It is assumed that $p(\mathbf{y}|\boldsymbol{\theta})$ satisfies the regularity condition (24). Now, consider a biased Bayesian estimator $\hat{\boldsymbol{\theta}}_B$ with bias $\mathbf{b}(\boldsymbol{\theta})$, then*

$$E_{\mathbf{y}}[(\hat{\boldsymbol{\theta}}_B - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}}_B - \boldsymbol{\theta})^{\mathsf{T}}] \geq \mathbf{H}(\boldsymbol{\theta})\mathbf{I}_B^{-1}(\boldsymbol{\theta})\mathbf{H}(\boldsymbol{\theta}) \quad (28)$$

where $\mathbf{H} = \mathbf{1} + \frac{\partial \mathbf{b}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \mathbf{b}(\boldsymbol{\theta}) \frac{\partial \ln p(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}^{\mathsf{T}}$. Since $\mathbf{H}(\boldsymbol{\theta})\mathbf{I}_B^{-1}(\boldsymbol{\theta})\mathbf{H}(\boldsymbol{\theta})$ is at least positive semidefinite, therefore we have the bound for the mse of each element of $\hat{\boldsymbol{\theta}}_B$ as

$$mse([\hat{\boldsymbol{\theta}}_B]_i) \geq \left[\mathbf{H}(\boldsymbol{\theta})\mathbf{I}_B^{-1}(\boldsymbol{\theta})\mathbf{H}(\boldsymbol{\theta})\right]_{ii} \quad i = 1, 2, \cdots, N. \quad (29)$$

3.3. Lower bounds in the presence of the nuisance parameters

In this section, we extend our results to the case where part of the unknows are nuisance parameters. In the presence of the nuisance parameters, it is desired to eliminate them by integrating them out from the posterior distribution and what is resulted is called the marginalized Bayesian estimator. To develop the lower bound on the variance of the marginalized estimators, we partition $\boldsymbol{\theta}$ as $\boldsymbol{\theta}^{\mathsf{T}} = [\boldsymbol{\phi}^{\mathsf{T}}, \boldsymbol{\psi}^{\mathsf{T}}]$, where $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ represent the subvector of the desired parameters and the nuisance parameters respectively. Now, suppose the prior distributions for nuisance parameters are $p(\Psi)$. The marginalized likelihood function can be thus defined as

$$p(\mathbf{y}|\boldsymbol{\phi}) = \int p(\mathbf{y}|\boldsymbol{\theta})p(\boldsymbol{\psi})d\boldsymbol{\psi}. \quad (30)$$

Then all the discussion in the previous sections can be directly applied to develop the lower bound for the Bayesian estimators of $\boldsymbol{\phi}$.

4. EXAMPLE

Let us consider a simple problem of estimation of DC level in the white Gaussian noise. The problem can be model as

$$y_t = A + w_t \qquad t = 0, 1, \cdots, N-1 \tag{31}$$

where y_t denotes the observation at time t, A is the unknown DC level, and $w_t \sim \mathcal{N}(0, \sigma^2)$ with σ^2 known. It is well known that the CRLB for the estimate of A is [2]

$$var(\hat{A}) \geq \frac{\sigma^2}{N} \tag{32}$$

and the efficient estimator is the sample mean estimator which is expressed as $\hat{A}_{ML} = \frac{1}{N}\sum_{t=0}^{N-1} y_t$.

Now, to study the problem from a Bayesian perspective, we adopt a conjugate prior for A, i.e., $p(A) = \mathcal{N}(\mu_A, \sigma_A^2)$ with μ_A and σ_A^2 preassigned. We know that the choice of μ_A and σ_A^2 will affect the performance of the Bayesian estimator. Typically, a more accurate estimator than $\hat{A}_{ML}$ could be obtained when μ_A is chosen around the true value of A, and performance of the Bayesian estimator would be more resemble to $\hat{A}_{ML}$ when $\sigma_A^2 \to \infty$. We thus should expect that the lower bound of the Bayesian estimator embodies above phenomenon. First, the PCRLB can be shown as

$$Bmse(\hat{A}) \geq (\frac{N}{\sigma^2} + \frac{1}{\sigma_A^2})^{-1}. \tag{33}$$

Obviously this bound is unable to reflect the effect of the prior $p(A)$ on the accuracy of the estimator. This is simply because A is taken as a random parameter. Next, the lower bound on the variance of any unbiased Bayesian esitmator is obtained from Theorem 1 as

$$var(\hat{A}_B) \geq (\frac{N}{\sigma^2} + \frac{(A-\mu_A)^2}{\sigma_A^4})^{-1} \tag{34}$$

Several observations arise on the bound. First, the bound depends on A. Second, no matter what the choices are for μ_A and σ_A^2, the bound is not higher than the CRLB. However, the bound still can not correctly reflect the the impact of $p(A)$ on the accuracy of the estimator. Actually, one can show that there does not exist an unbiased Bayesian estimator which attains the bound (34). In fact, the famous Bayesian estimators are often biased. For instance, the MMSE estimator of the problem is

$$\hat{A}_{MMSE} = \alpha \hat{A}_{ML} + (1-\alpha)\mu_A \tag{35}$$

where $\alpha = \sigma_A^2/(\sigma_A^2 + \sigma^2/N)$. This estimator is biased with a bias of $b(\hat{A}_{MMSE}) = (1-\alpha)(\mu_A - A)$. Besides, the variance of $\hat{A}_{MMSE}$ is $\alpha^2\sigma^2/N$. Now let us determine the lower bound on the MSE of all the Bayesian estimators which have the bias $b(A)$. It follows from Theorem 2 that this bound can be written as

$$\begin{aligned} mse(\hat{A}) &\geq \frac{(\alpha + (1-\alpha)(\mu_A - A)^2/\sigma_A^2)^2}{N/\sigma^2 + (A-\mu_A)^2/\sigma_A^4)} \\ &= \alpha^2\sigma^2/N + (1-\alpha)^2(A-\mu_A)^2 \\ &= b(\hat{A}_{MMSE}) + var(\hat{A}_{MMSE}) \\ &= mse(\hat{A}_{MMSE}) \end{aligned} \tag{36}$$

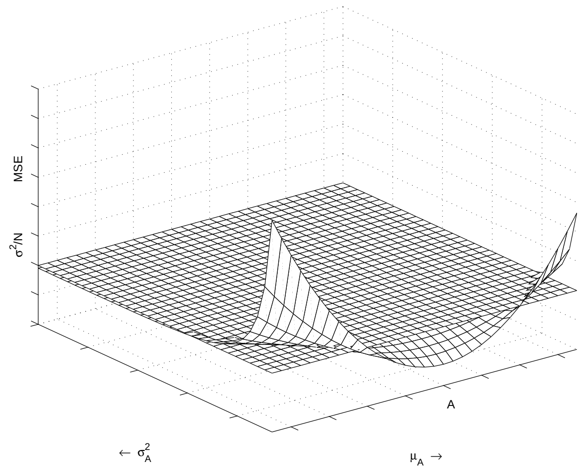

Figure 1: Plot of CRLB and the proposed bound for the example as a function of μ_A and σ_A.

The last equality indicates that among all the Bayesian estimators which bear the bias $b(\hat{A}_{MMSE})$, the MMSE estimator is the one which attains the bound. In Figure 1, the bound (36) is plotted together with the CRLB as a function of μ_A and σ_A. It shows that the bound (36) also correctly reflects the aforementioned phenomenon regarding the different choice of μ_A and σ_A on accuracy of the estimator. Obvious, the bound (36) is the only bound which can reflect the true state of the Bayesian estimators for deterministic parameters.

5. CONCLUSION

Lower bounds were developed for both unbiased and biased Bayesian estimators of deterministic parameters. We have shown that the variance of any unbiased Bayesian estimator which attains the proposed bound is always smaller than CRLB. Our example demonstrated that our proposed bounds are the appropriate benchmark on the performance of Bayesian esitmators for deterministic parameters.

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