ANALYTICAL REPRESENTATION FOR POSITIVE α-STABLE DENSITIES

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ABSTRACT

In this paper, we present an analytical approximation to positive α-stable probability distribution functions, which in general do not possess a compact analytical form. Our approximation is based on decomposing a positive α-stable random variable into a product of a Pearson and another positive stable random variable. This decomposition allows one to approximate any positive stable pdf as a mixture of Pearson densities, hence providing an analytical representation. This representation allows one to employ maximum likelihood estimation and Bayesian techniques in the presence of positive α-stable noise or signals. The efficiency of the decomposition is demonstrated by simulation studies.

1. INTRODUCTION

The α-stable distribution family has received great interest during the last few years in the signal processing community due to its success in modelling impulsive data. In particular, it has found applications in areas such as radar signal processing, financial time series modelling, telecommunications, and teletraffic modelling [1].

Despite its popularity as an impulsive noise model, the other dimension of α-stable distribution, that is its potential in modelling skewed (unsymmetric) data as well has been ignored other than a few recent work [2, 3, 4, 5, 6], and almost all work in the literature concentrated on symmetric α-stable ($S\alpha S$) distributions. However, many real data show unsymmetric characteristic which cannot be accommodated by the Gaussian or symmetric α-stable distributions. Examples include teletraffic data, financial time-series, geophysical signals and texture images. With the increasing interest especially in teletraffic analysis, one needs to address unsymmetric distributions as well, and skewed α-stable distributions provide a flexible framework for modelling such data.

The lack of an analytical expression for the α-stable pdf is a major drawback which prevents the utilisation of standard statistical signal processing techniques such as maximum likelihood estimation and Bayesian estimation. A partial remedy for the lack of a compact analytical form for the probability density function was brought by Kuruoglu et al. [7, 8] who suggested a Gaussian mixture representation for the symmetric α-stable case. Unfortunately, their representation does not generalise to the skewed case.

In this paper, we introduce a new analytical representation for a special case of the skewed α-stable pdf that is positive(or negative) α-stable distributions ($\beta = \mp 1, \alpha < 1$). The new model, which we believe is the first such model in the literature, is interesting in various aspects: it is not an ad-hoc model and is developed starting from some basic properties of the α-stable distribution, it is computationally efficient and is in the form of a mixture model which provides a simple parametrisation. The difference from the analytical model suggested for the symmetric α-stable case is that the components in the mixture are not Gaussian kernels but Pearson density functions.

The paper is organised as follows: Section 2 provides the definition and some basic properties of the α-stable distribution family which will be exploited in the following sections. Section 3 introduces the analytical model based on the Pearson distribution. Section 4 provides simulation studies using the new model. Section 5 concludes the text by a discussion of the applications of the new model.

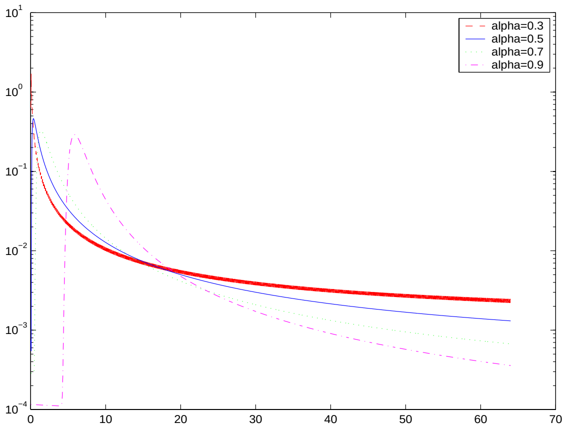

Fig. 1. Plots of positive α-stable pdfs for various values of α.

2. α-STABLE DISTRIBUTIONS

2.1. Definition

The α-stable distribution can be described most conveniently by its characteristic function due to the lack of a compact analytical expression for the pdf. The characteristic function is simply the Fourier transform of the pdf.

$$\varphi(t) = \begin{cases} \exp\{j\mu t - \gamma|t|^\alpha[1 + j\beta \mathrm{sign}(t)\tan(\frac{\alpha\pi}{2})]\}, \text{if } \alpha \neq 1 \\ \exp\{j\mu t - \gamma|t|^\alpha[1 + j\beta \mathrm{sign}(t)\frac{2}{\pi}\log|t|]\}, \text{if } \alpha = 1 \end{cases} \quad (1)$$

where α sets the degree of the impulsiveness of the distribution. Smaller values of α correspond to heavier tailed distributions and hence to more impulsive behaviour. β is the symmetry parameter that determines the skewness of the distribution. $\beta = 0$ corresponds to a symmetric distribution in which case the distribution is called symmetric α-stable ($S\alpha S$). γ, dispersion, is the scale parameter. μ is the location parameter. When $\beta = \mp 1$ the pdf is maximally skewed. In addition if $\alpha < 1$, the distribution is called positive (negative) stable and assumes only positive (negative) values. The Fourier transform of this expression leads to a compact form only in three special cases, namely the Gaussian distribution ($\alpha = 2$), the Cauchy distribution ($\alpha = 1, \beta = 0$) and the Pearson (or Lévy) distribution ($\alpha = 0.5, \beta = 1$).

In this paper, we are particularly interested in positive α-stable case, therefore we give the plots of several positive α-stable pdfs for various values of α (Fig. 1).

2.2. Decomposition into a Product of Two Other α-Stable Variables

One can decompose any α-stable random variable into a product of two other α-stable random variables which are specified by the following theorem.

Theorem 1: (Hardin [9])

Let Y and X be independent with $Y \sim S_{\alpha_y}(\gamma_y, 1, 0)$ and $X \sim S_{\alpha_x}(\gamma_x, \beta_x, 0)$ where $0 < \alpha_y < 1$, and $0 < \alpha_x \leq 2$, and $\alpha_x \neq 1 \neq \alpha_x\alpha_y$. Then, $Z = Y^{1/\alpha_x}X \sim S_{\alpha_z}(\gamma_z, \beta_z, 0)$ where $\alpha_z = \alpha_x\alpha_y$, $\beta_z = \frac{\tan(\alpha_y\theta)}{\tan(\alpha_y\alpha_x\pi/2)}$, $\gamma_z = \gamma_x^{\alpha_y}\gamma_y(1 + \beta_x^2\tan^2(\frac{\alpha_x\pi}{2}))^{\alpha_y/2}\frac{\cos(\alpha_y\theta)}{\cos\frac{\alpha_y\pi}{2}}$, and $\theta = \arctan(\beta_x \tan\frac{\alpha_x\pi}{2})$. In the case $\alpha_x < 1$, if $\beta_x \in -1, 0, 1$, then $\beta_z = \beta_x$.

Proof: See [9].

There are two important special cases of this theorem. The first one is when X is distributed with a Gaussian distribution ($\alpha_x = 2, \beta = 0$) in which case the equations simplify significantly and one obtaines the form given in [10], page 20. This special case is basically the statement of the fact that symmetric α-stable distributions are conditionally Gaussian. Motivated by this property, an analytical approximation was presented in [7, 8] based on scale mixtures of Gaussians.

3. SCALE MIXTURES OF PEARSONS

3.1. The $0 < \alpha < 1/2$ case

In this work, we consider another special case of Theorem 1, that is when X has a Pearson (or Lévy) distribution, $\alpha_x = 0.5, \beta_x = 1$. The pdf of the Pearson distribution is given by:

$$\left(\frac{\gamma}{2\pi}\right)^{1/2}\frac{1}{x^{3/2}}\exp\left(-\frac{\gamma}{2x}\right). \quad (2)$$

In this case Theorem 1 becomes:

Corollary 1.1: Let Y and X be independent with $Y \sim S_{\alpha_y}(\gamma_y, 1, 0)$ and $X \sim S_{0.5}(\gamma_x, 1, 0)$ where $\alpha_x \neq 2$. Then $Z = Y^2X \sim S_{\alpha_y/2}(\gamma_z, 1, 0)$ where $\gamma_z = 2^{\alpha_z}\gamma_x^{\alpha_y}\gamma_y\frac{\cos(\frac{\alpha_y\pi}{4})}{\cos(\frac{\alpha_y\pi}{2})}$.

For a given Z, the dispersion of Y is given by

$$\gamma_y = \frac{\gamma_z}{2^{\alpha_z}\gamma_x^{2\alpha_z}}\frac{\cos(\alpha_z\pi)}{\cos\left(\frac{\alpha_z\pi}{2}\right)}. \quad (3)$$

If additionally we choose $\gamma_x = 1$, this equation reduces to:

$$\gamma_y = \frac{\gamma_z}{2^{\alpha_z}}\frac{\cos(\alpha_z\pi)}{\cos\left(\frac{\alpha_z\pi}{2}\right)}. \quad (4)$$

Now, call $V = Y^2$, then from pdf conversion formula given by

Theorem 2: ([11])

$$\text{If V=g(Y), then } f_V(v) = f_Y(y)|_{g^{-1}(v)} \left|\frac{dy}{dv}\right|, \qquad (5)$$

we obtain

$$f_V(v) = f_Y(\sqrt{v})\frac{1}{2\sqrt{v}}. \qquad (6)$$

Next we would like to express the pdf of Z in terms of the pdfs of X and Y.

Theorem 3:

If $Z = VX$, then

$$f_Z(z) = \int_0^\infty \frac{1}{v} f_X\left(\frac{z}{v}\right) f_V(v) dv \qquad (7)$$

Proof: See [12], [11].

Substituting $f_V(v)$ from Eq. (6), we obtain:

$$f_Z(z) = \frac{1}{2}\int_0^\infty \frac{1}{v^{3/2}} f_X\left(\frac{z}{v}\right) f_Y(\sqrt{v}) dv. \qquad (8)$$

This is a scale mixture of Pearsons representation and can be used to obtain great computational advantage in Bayesian estimation problems as discussed later in the paper.

We can also discretize this integral uniformly to obtain a finite mixture of Pearsons model

$$f_Z(z) = \frac{1}{K}\sum_v \frac{1}{v^{3/2}} f_Y(\sqrt{v}) f_X\left(\frac{z}{v}\right) \qquad (9)$$

where K is a normalisation factor depending on the sampling period.

3.2. The $1/2 < \alpha < 1$ case

The approximation presented above provides a representation only for the case $0 < \alpha_z < 1/2$ since the expressions are valid only for $0 < \alpha_y < 1$ and $\alpha_y = 2\alpha_z$. Ideally one would like to derive similar decompositions for all values of α_z that is $0 < \alpha_z < 2$. Towards this end we suggest a modification in the above derivation: instead of X, choose Z to follow a Pearson distribution. In that case the new product decomposition can be envisaged as

$$X = \frac{1}{Y^{1/\alpha_x}} Z \qquad (10)$$

In that case Corollary 1.1 is modified into:

Corollary 1.2.: Let Y and X be independent and $Y \sim S_{\alpha_y}(\gamma_y, 1, 0)$ and $X \sim S_{1/2\alpha_y}(\gamma_x, 1, 0)$ where

$$\gamma_y = \sqrt{2}\cos\left(\frac{\pi}{4\alpha_x}\right)\cos\left(\frac{\alpha_x \pi}{2}\right)^{1/2\alpha_x} \gamma_x^{-1/2\alpha_x}, \qquad (11)$$

$0 < \alpha_y < 1$ and $0 < \alpha_x < 1$. Then, $Z \sim S_{0.5}(1, 1, 0)$.

We proceed in a manner similar to the derivation in the previous subsection: Call $V = \frac{1}{Y^{1/\alpha_x}}$, then from Eq. (6),

$$f_V(v) = \alpha_x f_Y(v^{-\alpha_x}) v^{-\alpha_x - 1}. \qquad (12)$$

Similar to the Eq. (7), we have

$$f_X(x) = \int_0^\infty \frac{1}{v} f_Z\left(\frac{x}{v}\right) f_V(v) dv. \qquad (13)$$

Substituting in the above expression for $f_V(v)$, we obtain:

$$f_X(x) = \alpha_x \int_0^\infty v^{-\alpha_x - 2} f_Z\left(\frac{x}{v}\right) f_Y\left(v^{-\alpha_x}\right) dv. \qquad (14)$$

This is yet another scale mixture of Pearsons model. One can discretize to obtain a finite mixture of Pearsons model:

$$f_X(x) = \frac{1}{K}\sum_v v^{-\alpha_x - 2} f_Y\left(v^{-\alpha_x}\right) f_Z\left(\frac{x}{v}\right) \qquad (15)$$

where K is a normalisation factor.

Note that this derivation is not valid for $1 < \alpha_x < 2$ for which case the expression for β_x from Theorem 1 leads to a term coupled to α_x and the theorem looses generality. Therefore the Pearson mixture model cannot be generalised to positive α-stable distributions other than the positive stable case.

4. SIMULATION RESULTS

To demonstrate the efficiency of the approximation, we obtained a Pearson mixture representation for a positive stable pdf with $\alpha = 0.4, \beta = 1, \gamma = 1$. X in the decomposition is a Pearson random variable ($X \sim S_{0.5}(1, 1, 0)$) and $Y \sim S_{0.4}(0.289, 1, 0)$. In Fig. 2, we present the plot of both the actual pdf (obtained by numerical integration) and of the Pearson mixture approximation. The approximation is very good: it is difficult to differentiate the two curves.

In the simulation only 1000 data points were used and the result was obtained in less than one second using Matlab on a 2 GHz processor although the code was not optimised.

5. APPLICATIONS AND CONCLUSIONS

In this paper, we introduced a new analytical representation for a special case of the skewed α-stable distribution, when $\beta = \mp 1$ and $\alpha < 1$. To the best of our knowledge, this is the first such representation suggested in the literature. The model is computationally

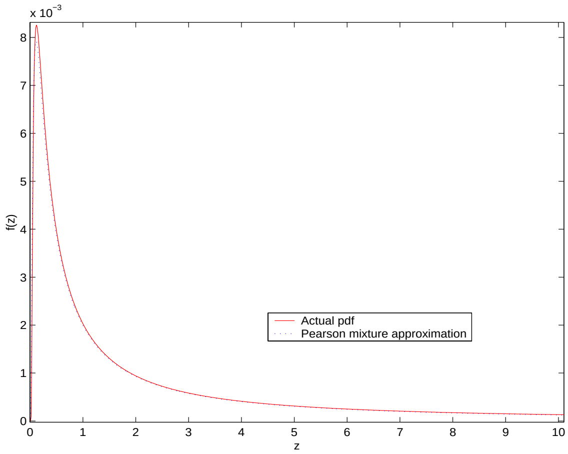

Fig. 2. Actual pdf and the Pearson mixture approximation to it. $\alpha = 0.4, \gamma = 1$.

efficient, it does not require costly algorithms such as the expectation-maximization algorithm or Bayesian estimation. The model is also motivated directly by some basic properties of α-stable distributions and therefore is not ad-hoc. The simulation studies demonstrate the success of the new representation.

The positive stable distribution, despite recent interest in it [2] is not easily incorporated in inference schemes because of the non-analytical form of the pdf which will prevent the likelihood functions from being evaluated. With the new representation, it is now possible to employ Bayesian and maximum likelihood estimation techniques in problems involving skewed noise or signals that can be modelled with a positive stable distribution. Using the product decomposition into scale mixtures of Pearson's, one can obtain the following generation mechanism

$$z_t \sim P(y_t \gamma), \qquad y_t \sim S_{2\alpha}(\gamma_y, 1, 0) \tag{16}$$

where $P(.)$ is the Pearson pdf. This inference mechanism, as described in [13], would aid in Markov chain Monte Carlo (MCMC) and expectation-maximization (EM) methods involving positive α-stable distributions.

The model also has potential applications in detection schemes which employ the pdf of the noise for the design of receiver nonlinearities.

6. REFERENCES

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