HARMONIC SIGNAL ANALYSIS WITH KALMAN FILTERS USING PERIODIC ORBITS OF NONLINEAR ODEs

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ABSTRACT

This paper suggests a new approach to the problem of harmonic signal estimation. The idea is to model the harmonic signal as a function of the state of a second order nonlinear ordinary differential equation (ODE). The function of the right hand side of the nonlinear ODE is parameterized with a polynomial model. A Kalman filter and an extended Kalman filter are then developed. The proposed methodology reduces the number of estimated unknowns in cases where the actual signal generation resembles that of the imposed model. This is expected to result in an improved accuracy of the estimated parameters, as compared to existing methods.

1. INTRODUCTION

The modeling of signals with harmonic spectra has widespread applications. Examples include vibration analysis and overtone analysis in power networks. The measurement of linearity of electronic power amplifiers and other devices using sinusoidal inputs is another very important field of application.

One of the most widespread signal processing methods ever, the periodogram method in combination with fast Fourier transform (FFT) techniques, forms a baseline against which harmonic signal analysis methods can be compared. See [1] for detailed algorithms and performance analysis issues. Parametric methods for line spectra are directly applicable to the harmonic signal estimation problem. The references [1]-[3] discuss a number of such methods. Many theoretical results on expected performance are available in [1]-[4].

The present paper is inspired by a possible model for the generation of periodic signals, namely nonlinear ordinary differential equations (ODEs). There is a rich theory on the subject as outlined in e.g. [5]. Some of the strongest results of the theory concern ODEs with two state variables. There are several powerful theorems on the existence of periodic solutions to ODEs in R^2 - hence it seems to be advantageous to base estimation algorithms on second order ODEs.

The signal model used in the paper is obtained by introducing a polynomial parameterization of the right hand side of a general second order ODE, and by defining the harmonic signal to be modeled as a function of the states of this ODE. A Kalman filter algorithm [6] and a nonlinear approach using the extended Kalman filter [6] are then described. Constraints are imposed on the model structure to reduce the number of parameters. It is shown that these constraints are valid for a very general class of second order differential equations. Furthermore, as compared to [7], the algorithms of the present paper are recursive and less sensitive to noise.

What are the advantages offered by the approach taken? First, many systems that generate harmonic signals are best described by nonlinear ODEs. Examples include tunnel diodes, pendulums, biological predator-prey systems and radio frequency synthesizers, see [5]. Many of these systems are described by second order ODEs with polynomial right hand sides, and it can be expected that there are then good opportunities to obtain highly accurate models by estimating only a few parameters. This implies that the achievable accuracy would be improved, as compared to methods that do not impose the same amount of prior information.

The paper is organized as follows. Section 2 introduces the details on the model, including a definition of the parameterization. Section 3 discusses algorithms, while section 4 presents a simulations. Conclusions appear in section 5.

2. THE ODE MODEL AND ITS PARAMETERIZATION

2.1. Measurements and modeled signals

The starting point is the *discrete time* measured signal

$$z(t) = y(t) + e(t). \qquad (1)$$

Here $y(t)$ is the *continuous time* signal to be modeled and $e(t)$ is the discrete time measurement. The condition

C1) $y(t+T) = y(t)$, $\forall t \in R$, $0 < T < \infty$,

means that $y(t)$ is assumed to be periodic. Furthermore, $e(t)$ is assumed to be zero mean Gaussian white noise, i.e.

C2) $e(t) \in N(0, s^2)$, $E[e(t)e(t+kT_s)] = d_{k,0} s^2$.

T_S denotes the sampling period.

2.2. Model structure and parameterization

The main idea of the paper is to model the generation of the signal $y(t)$ by means of an ordinary differential equation of order two. A general model structure of this kind is

$$\begin{pmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{pmatrix} = \begin{pmatrix} f_1(x_1(t), x_2(t), q_1) \\ f_2(x_1(t), x_2(t), q_2) \end{pmatrix} \qquad (2)$$

$$z(t) = (c_1 \quad c_2)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + e(t). \qquad (3)$$

In (2) and (3) $(x_1(t) \quad x_2(t))^T$ is the state vector, $(c_1 \quad c_2)$ is the vector with the *selected* output weighting factors and

$$q = (q_1^T \quad q_2^T)^T \qquad (4)$$

is the unknown parameter vector. Note that (2) is given in continuous time, while (3) is a discrete time equation.

Additional structure will now be imposed on the model by choosing the state variables and the output relation as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ dy/dt \end{pmatrix}. \qquad (5)$$

$$(c_1 \quad c_2) = (1 \quad 0), \qquad (6)$$

referring to the underlying second order ODE

$$\frac{d^2y}{dt^2} = f_2\left(y, \frac{dy}{dt}, q_2\right) \qquad (7)$$

Note that *all systems* where the measured quantity can be modeled as (7) can be written as (5) and (6). This has the advantage of reducing the number of parameters since $dx_1/dt = x_2$ and only $f_2(x_1(t), x_2(t), q_2)$ of (2) needs to be parameterized. Most often conventional physical modeling results in ODEs where the measured quantity is the dependent variable, a fact that motivates the selection. Hence, the constrained structure is believed to be a quite general one.

As stated in the introduction, the parameterization that is used in this paper is chosen to be polynomial. Together with (5) - (7), this choice gives

$$\begin{aligned} f_1(x_1(t), x_2(t), q_1) &= x_2(t) \\ f_2(x_1(t), x_2(t), q_2) &= \sum_{l=0}^{L_2}\sum_{m=0}^{M_2} q_{2,l,m} x_1^l(t) x_2^m(t). \end{aligned} \qquad (8)$$

2.3. Discretization

In order to formulate complete discrete time models, the continuous time ODE model (2) needs to be discretized in time. This is done by exploiting an Euler forward numerical integration scheme. For simplicity, the discretization interval is selected to be equal to the sampling period T_S. The result is

$$\begin{aligned} x_1(t+T_S) &= x_1(t) + T_S x_2(t) \\ x_2(t+T_S) &= x_2(t) + T_S \sum_{l=0}^{L_2}\sum_{m=0}^{M_2} q_{2,l,m} x_1^l(t) x_2^m(t). \end{aligned} \qquad (9)$$

The model (9) can be compactly written as

$$\begin{aligned} x_1(t+T_S) - x_1(t) &= T_S x_2(t) \\ x_2(t+T_S) - x_2(t) &= j_2^T(x_1(t), x_2(t)) q_2 \end{aligned} \qquad (10)$$

$$\frac{1}{T_S} j_2(x_1(t), x_2(t)) = \left(1 \quad \ldots \quad x_2^{M_2}(t) \quad \ldots \quad x_1^{L_2}(t) \quad \ldots \quad x_1^{L_2}(t) x_2^{M_2}(t)\right)^T \qquad (11)$$

$$q_2 = \left(q_{2,0,0} \quad \ldots \quad q_{2,0,M_2} \quad \ldots \quad q_{2,L_2,0} \quad \ldots \quad q_{2,L_2,M_2}\right)^T. \qquad (12)$$

3. ALGORITHMS

3.1. The Kalman Filter

The Kalman filter is obtained by introduction of the parameters as states, using a random walk assumption. Other alternatives than a random walk assumption are possible. However the choice used here is standard in the literature. Note that the construction of the regression vector $j_2(x_1(t), x_2(t))$ of (11) is based on the exact states - a requirement that cannot be met. Two ideas to resolve this problem are presented in this paper.

The idea of the Kalman filter algorithm is to replace the first state appearing in the regression vector by the measured output signal, a choice that follows from the definition of the output vector in (6). Because of the selected structure, the second state is replaced by an estimate of the derivative of the measured output signal. This latter signal, denoted $(dz(t)/dt)_{est}$, is obtained from a differentiating filter.

The state vector of the Kalman filter is now augmented with the estimated parameter vector as follows

$$\mathbf{x}(t) = \left(x_1(t) \quad x_2(t) \quad q_2^T(t)\right)^T \qquad (13)$$

Together with (6), (9) - (12), this results in

$$\mathbf{x}(t+T_S)=\mathbf{F}\left(T_S,z(t),\left(dz(t)/dt\right)_{est}\right)\mathbf{x}(t)+\mathbf{w}(t)$$
$$z(t)=\mathbf{Hx}(t)+e(t) \qquad (14)$$

$$\mathbf{F}\left(T_S,z(t),\left(dz(t)/dt\right)_{est}\right)=\begin{pmatrix}1 & T_S & \mathbf{0}^T \\ 0 & 1 & j_2^T\left(z(t),\left(dz(t)/dt\right)_{est}\right) \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{(M_2+1)(L_2+1)}\end{pmatrix} \qquad (15)$$

$$\mathbf{H}=\begin{pmatrix}1 & 0 & \mathbf{0}^T\end{pmatrix}. \qquad (16)$$

In order to complete the Kalman filter description, noise properties and initial values need to be set. The measurement noise properties follow from C2). Because of the nonlinear data dependence in $j_2^T\left(z(t),\left(dz(t)/dt\right)_{est}\right)$, the system noise is not Gaussian. However, the Kalman filter can be applied anyway. Provided that C3) and C4) below hold with a sufficient accuracy, the performance of the algorithm can be expected to be close to that of an exact Kalman filter.

C3) $\mathbf{w}(t)=\begin{pmatrix}w_1(t) & w_2(t) & \mathbf{w}_{q_2}^T(t)\end{pmatrix}^T \in N(\mathbf{0},\mathbf{R}_1)$ and

$E\left[\mathbf{w}(t)\mathbf{w}^T(t+kT_S)\right]=d_{k,0}\mathbf{R}_1$.

C4) The initial values $\hat{\mathbf{x}}(t_0\,|\,t_0-T_S)=E[\mathbf{x}(t_0)\,|\,\mathbf{x}(t_0-T_S)]$ and $\mathbf{P}(t_0\,|\,t_0-T_S)=E\left[(\mathbf{x}(t_0)-\hat{\mathbf{x}}(t_0\,|\,t_0-T_S))(\mathbf{x}(t_0)-\hat{\mathbf{x}}(t_0\,|\,t_0-T_S))^T\right]$ define a Gaussian distribution of the prior state.

The Kalman filter is now given by the same equations as the extended Kalman filter, see (19) below, replacing the linearized parts with the matrices described by (14) – (16).

3.2. The Extended Kalman Filter

The extended Kalman filter differs from the Kalman filter in one crucial point - the state propagation matrix is built up from the estimated states rather than directly from measured data as shown in (17), (18). These states are replaced by estimated states in the algorithm, as shown in the three last lines of (19).

$$\mathbf{x}(t+T_S)=\mathbf{F}(T_S,x_1(t),x_2(t))\mathbf{x}(t)+\mathbf{w}(t) \qquad (17)$$
$$z(t)=\mathbf{Hx}(t)+e(t)$$

$$\mathbf{F}(T_S,x_1(t),x_2(t))=\begin{pmatrix}1 & T_S & \mathbf{0}^T \\ 0 & 1 & j_2^T(x_1(t),x_2(t)) \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{(M_2+1)(L_2+1)}\end{pmatrix}. \qquad (18)$$

The equations (13) and (16) remain unaltered. Following [8] the extended Kalman filter recursions are now given by

$$\mathbf{K}(t)=\mathbf{P}(t|t-T_S)\mathbf{H}^T\left(\mathbf{HP}(t|t-T_S)\mathbf{H}^T+R_2\right)^{-1}$$
$$\hat{\mathbf{x}}(t\,|\,t)=\hat{\mathbf{x}}(t\,|\,t-T_S)+\mathbf{K}(t)(z(t)-\mathbf{H}\hat{\mathbf{x}}(t\,|\,t-T_S))$$
$$\mathbf{P}(t|t)=\mathbf{P}(t|t-T_S)-\mathbf{P}(t|t-T_S)\mathbf{H}^T$$
$$\times\left(\mathbf{HP}(t|t-T_S)\mathbf{H}^T+R_2\right)^{-1}\mathbf{HP}(t|t-T_S)$$
$$\hat{\mathbf{x}}(t+T_S\,|\,t)=\mathbf{F}(T_S,\hat{\mathbf{x}}(t\,|\,t))\hat{\mathbf{x}}(t\,|\,t)$$
$$\tilde{\mathbf{F}}(t)=\left.\frac{¶\,(\mathbf{F}(T_S,\mathbf{x})\mathbf{x})}{¶\mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}(t|t)}$$
$$\mathbf{P}(t+T_S\,|\,t)=\tilde{\mathbf{F}}(t)\mathbf{P}(t\,|\,t)\tilde{\mathbf{F}}^T(t)+\mathbf{R}_1. \qquad (19)$$

Remark 1: The modifications as compared to the Kalman filter may seem minor. However, it is stressed that *they are not.* First, since there are no nonlinear transformations of noisy measurements, C3 can now be expected to hold exactly, and no significant bias problems are expected. Secondly, the dynamics of (17) and (18) is *highly nonlinear*. It is in fact polynomial. Hence instability phenomena and even finite escape time effects may come into play. See [5] for further details.

4. SIMULATION STUDY

The proposed algorithms were tested with data generated by a Van der Pol oscillator as described in [5]. This systems can here be described by the ODE

$$\begin{pmatrix}\dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt}\end{pmatrix}=\begin{pmatrix}x_2 \\ -x_1+2\left(1-x_1^2\right)x_2\end{pmatrix}. \qquad (20)$$

Data were generated by solving (20) with the MATLAB™ routine ode45, using T_S=0.010. Initial states were chosen as $\begin{pmatrix}x_1(0) & x_2(0)\end{pmatrix}^T=\begin{pmatrix}0.000 & 1.000\end{pmatrix}^T$. White Gaussian noise was added to $x_1(t)$ to obtain data with a signal to noise ratio of 15 dB. The algorithms were initialized with $\begin{pmatrix}\hat{x}_1(0\,|-T_S) & \hat{x}_2(0\,|-T_S) & \hat{q}_2^T(0\,|-T_S)\end{pmatrix}^T=\begin{pmatrix}-0.500 & 0.500 & \mathbf{0}^T\end{pmatrix}^T$ and the remaining variables were selected as $\mathbf{P}(0|-T_S)=10\mathbf{I}$, $\mathbf{R}_1=10^{-5}\,\mathbf{I}$ and $R_2=1$. The parameter vector was selected according to (9) and (12) with $L_2=M_2=2$. The differentiating filter was selected as a simple finite difference. The results are available in Fig. 1 - Fig. 3, where the Kalman filter and the extended Kalman filter are compared.

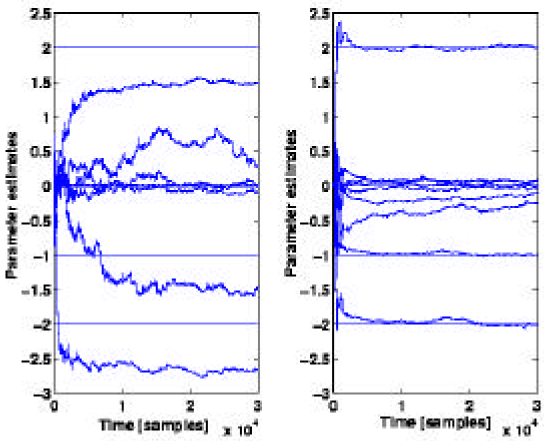

Figure 1: Parameter convergence. Kalman filter (left) and extended Kalman filter (right).

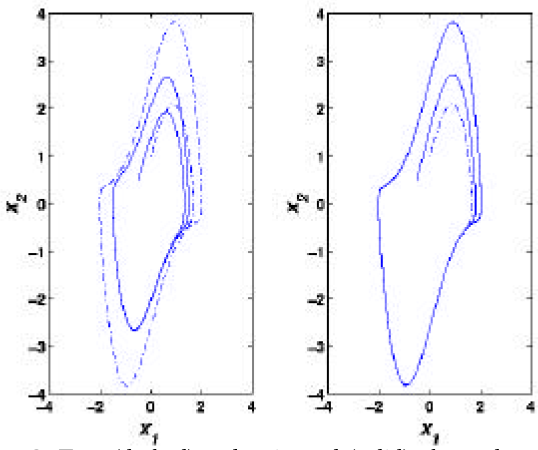

Figure2: True (dashed) and estimated (solid) phase plane plots. Kalman filter (left) and extended Kalman filter (right).

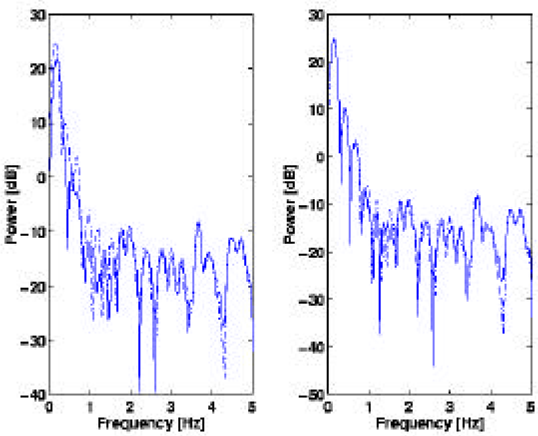

Figure 3: True (dashed) and estimated (solid) spectra. Kalman filter (left) and extended Kalman filter (right).

It can be concluded that the extended Kalman filter has a superior performance. It appears to be unbiased, which is not true for the Kalman filter. However, when a polynomial degree of three was tried, the extended Kalman filter became unstable at 15 dB SNR, while the Kalman filter still delivered usable results. For higher SNRs the extended Kalman filter again performed the best. This is in line with the predictions of section 3.

5. CONCLUSIONS

The paper has presented a novel approach to the modeling of harmonic signals. The main idea is to model the harmonic signal with a second order nonlinear ordinary differential equation with periodic orbits. Two algorithms were derived using this idea. The Kalman filter, although biased, performed robustly and can be recommended e.g. for initial value generation for the extended Kalman filter. The latter algorithm showed a performance that was superior to the Kalman filter.

Many open topics for further research exist. An analysis of the asymptotic performance of the proposed methods seems worthwhile. There are also fundamental issues related to the problem setting. When is it e.g. possible to approximate any periodic signal arbitrarily well by making a long enough expansion of the right hand side of a second order ODE?

6. REFERENCES

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