A MULTI-WINDOW FRACTIONAL EVOLUTIONARY SPECTRAL ANALYSIS

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ABSTRACT

In this work, we present a multiple window Evolutionary Spectral analysis on a non-rectangular time-frequency lattice based on a discrete fractional Gabor expansion. The traditional Gabor expansion uses a fixed, and rectangular time-frequency plane tiling. Many of the practical signals such as speech, music, etc., require a more flexible, non-rectangular time-frequency lattice for a compact representation. The proposed method uses a set of basis functions that are related to the fractional Fourier basis and generate a parallelogram-shaped tiling. Simulation results are given to illustrate the performance of our algorithm.

1. INTRODUCTION

Time-frequency (TF) analysis provides a characterization of signals in terms of joint time and frequency content [1]. Evolutionary Spectrum (ES) is one of the TF analysis methods [2] which is based on the decomposition of signals into sinusoids with random and time-varying amplitudes. In our previous work, we present a method to estimate the ES of discrete-time, non-stationary signals using Gabor expansions [3]. The Gabor expansion is a TF decomposition which represents a signal in terms of time and frequency translated basis functions [4]. Gabor basis functions are obtained by shifting and modulating with a sinusoid a single window function, which results in a fixed and rectangular TF plane tiling. However, many of the practical signals such as speech, music, biological, and seismic signals have time-varying frequency nature that is not appropriate for this type of analysis [5, 6]. Thus the Gabor expansion of such signals will require large number of coefficients yielding a poor TF localization [7, 8]. Therefore, the ES estimate we obtain by using the Gabor expansion suffers from a TF resolution problem. Here we present an ES analysis method based on a new, fractional Gabor expansion that uses a more flexible, parallelogram-shaped TF lattice [9]. The basis functions of the proposed expansion are related to the fractional Fourier series basis [10].

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2. ES ANALYSIS BASED ON GABOR EXPANSIONS

In [3], we present an Evolutionary Spectral (ES) estimate based on a multi-window Gabor expansion. In the following, we briefly present the discrete multi-window Gabor expansion, and our ES estimation.

2.1. A Multi-window Gabor Expansion

Multi-window Gabor expansion [3] represents a signal in terms of scaled and time and frequency shifted basis functions and is given for a finite-support signal $x(n),\ 0 \leq n \leq N-1$ by

$$x(n) = \frac{1}{I}\sum_{i=0}^{I-1}\sum_{m=0}^{M-1}\sum_{k=0}^{K-1} a_{i,m,k}\ \tilde{g}_{i,m,k}(n) \qquad (1)$$

where the basis functions are

$$\tilde{g}_{i,m,k}(n) = \tilde{g}_i(n-mL)e^{j\omega_k n} \qquad (2)$$

and $\omega_k = 2\pi kL'/N$. Here the synthesis window $\tilde{g}_i(n)$ is a periodic version by N of a time-scaled Gabor window as $g_i(n) = 2^{i/2}\ g(2^i n),\ i = 0, 1, ..., I-1$, [4] I is the number of windows used and Gabor expansion parameters M, K, L, and L' are positive integers constrained by $ML = KL' = N$ where M and K are the number of analysis samples in time and frequency, respectively. The Gabor coefficients can be calculated by

$$a_{i,m,k} = \sum_{n=0}^{N-1} x(n)\ \tilde{\gamma}^*_{i,m,k}(n) \qquad (3)$$

where $\tilde{\gamma}_{i,m,k}(n) = \tilde{\gamma}_i(n-mL)\ e^{j\frac{2\pi k}{K}n}$ and $\tilde{\gamma}_i(n)$ analysis window is solved from the biorthogonality condition between $\tilde{g}_i(n)$ and $\tilde{\gamma}_i(n)$ [4]. Gabor basis $\{\tilde{\gamma}_{i,m,k}(n)\}$ with fixed window and sinusoidal modulation tiles the TF plane in a rectangular fashion. Such methods usually provide signal representations with poor TF resolution [5].

2.2. Evolutionary Spectral Analysis

We obtain in [3] an evolutionary spectral estimate based on the Gabor coefficients. We consider the following discrete-time, discrete-frequency model for finite-extent, deterministic signals that is analogous to the Wold-Cramer representation of non-stationary random signals:

$$x(n) = \sum_{k=0}^{K-1} A(n,\omega_k) e^{j\omega_k n}, \qquad 0 \le n \le N-1, \quad (4)$$

where $\omega_k = 2\pi k/K$, and $A(n,\omega_k)$ is a time-frequency kernel. Comparing the two representations of $x(n)$ in (1) and (4), we have that the kernel is

$$\begin{aligned} A(n,\omega_k) &= \frac{1}{I}\sum_{i=0}^{I-1}\sum_{m=0}^{M-1} a_{i,m,k}\ \tilde{g}_i(n-mL) \\ &= \sum_{\ell=0}^{N-1} x(\ell)\ \mathrm{w}(n,\ell)\ e^{-j\omega_k \ell} \quad (5) \end{aligned}$$

where we substituted for the coefficients $\{a_{i,m,k}\}$ and defined the time-varying window

$$\mathrm{w}(n,\ell) = \frac{1}{I}\sum_{i=0}^{I-1}\sum_{m=0}^{M-1} \tilde{\gamma}_i^*(\ell-mL)\tilde{g}_i(n-mL). \quad (6)$$

Then the evolutionary spectrum of $x(n)$ is obtained by

$$S_{ES}(n,\omega_k) = \frac{1}{K}\ |A(n,\omega_k)|^2, \quad (7)$$

where the factor $1/K$ is used for proper energy normalization. The above ES analysis method with a fixed, and rectangular TF lattice yields a poor resolution. Several approaches have been proposed to improve the resolution of such sinusoidal representations: some of them are averaging estimates obtained using different windows [3], and maximizing energy concentration measures [5, 7]. In recent works, representations on a non-rectangular TF grid has attracted a considerable attention [6, 9]. A non-rectangular lattice is more appropriate for the TF analysis of signals with time-varying frequency content.

3. DISCRETE FRACTIONAL GABOR EXPANSION

We define a discrete fractional multi-window Gabor expansion for $x(n),\ 0 \le n \le N-1$, as follows:

$$x(n) = \frac{1}{I}\sum_{i=0}^{I-1}\sum_{m=0}^{M-1}\sum_{k=0}^{K-1} a_{i,m,k,\alpha}\ \tilde{g}_{i,m,k,\alpha}(n) \quad (8)$$

where $a_{i,m,k,\alpha}$ are the fractional Gabor coefficients, α is the order of the fraction, and the basis functions are

$$\tilde{g}_{i,m,k,\alpha}(n) = \tilde{g}_i(n-mL)\ W_{\alpha,k}(n)$$

Here $\tilde{g}_i(n)$ is defined as above, and $W_{\alpha,k}(n)$ is the fractional kernel,

$$W_{\alpha,k}(n) = e^{j[-\frac{1}{2}(n^2+(\omega_k \sin\alpha)^2)\cot\alpha + \omega_k n]}$$

which is similar to the Fractional Fourier Series basis functions presented in [10], and again $\omega_k = 2\pi k/K$. Above basis functions with this fractional kernel generate a parallelogram shaped TF sampling geometry. The expansion in (8) reduces to the above sinusoidal Gabor expansion for $\alpha = \pi/2$. The parameters M, K, L, and L', are same as in the traditional Gabor expansion. For numerically stable solutions we need that $L \le K$. The case where $L = K$ is called the critical sampling, and the case where $L < K$ is the oversampling. In our derivations, we always consider the general, oversampled case. The Gabor coefficients can be evaluated as before by

$$a_{i,m,k,\alpha} = \sum_{n=0}^{N-1} x(n)\ \tilde{\gamma}^*_{i,m,k,\alpha}(n) \quad (9)$$

where the analysis functions are

$$\tilde{\gamma}_{i,m,k,\alpha}(n) = \tilde{\gamma}_i(n-mL)\ W_{\alpha,k}(n)$$

and $\tilde{\gamma}_i(n)$ is periodic version of a $\gamma_i(n)$ that is solved from a fractional biorthogonality condition between $g_i(n)$ and $\gamma_i(n)$.

To obtain the completeness condition for the fractional Gabor basis for scale i, substitute (9) in (8):

$$\begin{aligned} x(n) &= \sum_{m=0}^{M-1}\sum_{k=0}^{K-1}\left(\sum_{\ell=0}^{N-1} x(\ell)\tilde{\gamma}_i^*(\ell-mL)W^*_{\alpha,k}(\ell)\right) \\ &\times\ \ \tilde{g}_i(n-mL)W_{\alpha,k}(n) \\ &= \sum_{\ell=0}^{N-1}\sum_{m=0}^{M-1}\sum_{k=0}^{K-1} \tilde{g}_i(n-mL)\tilde{\gamma}_i^*(\ell-mL) \\ &\times\ \ e^{j[-\frac{1}{2}(n^2-\ell^2)\cot\alpha + \omega_k(n-\ell)]} \end{aligned}$$

From the above equation, we obtain the completeness relation for basis $\{\tilde{g}_{i,m,k,\alpha}(n)\}$ as

$$\begin{aligned} \sum_{m=0}^{M-1}\sum_{k=0}^{K-1} & \tilde{g}_i(n-mL)\tilde{\gamma}_i^*(\ell-mL)e^{j[-\frac{1}{2}(n^2-l^2)\cot\alpha]} \\ & \times e^{j\ \omega_k(n-\ell)} = \delta(n-\ell) \quad (10) \end{aligned}$$

The fractional biorthogonality condition that we need to solve the analysis or dual function $\gamma_i(n)$ is obtained from the above completeness relation using a discrete Poisson sum formula as:

$$\begin{aligned} &\sum_{n=0}^{N-1} \tilde{g}_i^*(n+mK)e^{jk\frac{2\pi}{L}(n+mK)}\tilde{\gamma}_i(n) \\ &\times e^{j(nmK+\frac{m^2K^2}{2})\cot\alpha} = \frac{L}{K}\ \delta_m\delta_k \\ &0 \le m \le L'-1, \quad 0 \le k \le L-1 \quad (11) \end{aligned}$$

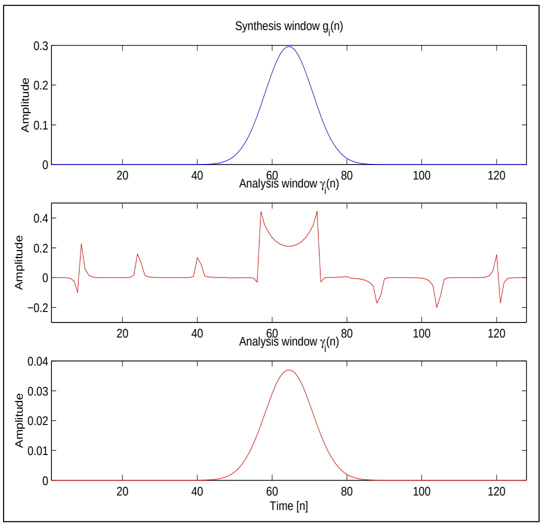

Fig. 1. A Gauss synthesis window (top figure), and its biorthogonal windows in critical (middle) and oversampling (bottom) cases.

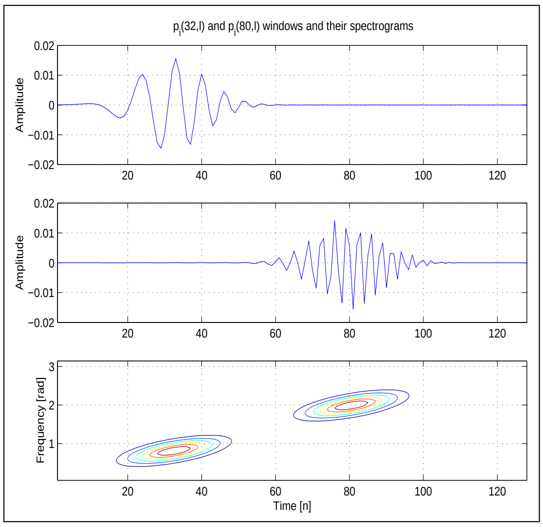

Fig. 2. The time–varying chirp window used in ES for $n = 32$ (top figure), $n = 80$ (middle) and their spectrogram (bottom).

Completeness and biorthogonality conditions given in equations (10) and (11) reduce to the conditions in the traditional case [4] for $\alpha = \pi/2$. This indicates that the above fractional expansion is the generalization of the Gabor expansion. In Fig. 1, we show a Gauss window $g_i(n), n = 0, 1, \cdots, 127$ on the top figure, and its biorthogonal $\gamma_i(n)$ for two different set of sampling parameters obtained by solving equation (11) for $\alpha = \pi/4$. The window in the middle is obtained using $L = 16, K = 16$ that is the critical sampling. The window at the bottom is calculated with $L = 8, K = 64$ as an example of the oversampling.

4. FRACTIONAL EVOLUTIONARY SPECTRAL ANALYSIS

In this section we present a fractional evolutionary spectral analysis method based on the above Gabor expansion. Here we consider the discrete–time, and discrete–frequency representation for $x(n)$ given in equation (1). Comparing this with the fractional Gabor representation in (8), we get the time–frequency kernel using window $\tilde{g}_i(n)$ as

$$A(n,\omega_k) = \frac{1}{I}\sum_{i=0}^{I-1}\sum_{m=0}^{M-1} a_{i,m,k,\alpha}\, \tilde{g}_i(n-mL) \times e^{-j\frac{1}{2}(n^2+(\omega_k \sin\alpha)^2)\cot\alpha} \qquad (12)$$

After replacing for the coefficients in (9) we have that

$$\begin{aligned} A(n,\omega_k) &= \sum_{\ell=0}^{N-1} x(\ell)\,\frac{1}{I}\sum_{i=0}^{I-1} \mathrm{p}_i(n,\ell)\; e^{-j\omega_k \ell} \\ &= \frac{1}{I}\sum_{i=0}^{I-1} A_i(n,\omega_k) \qquad (13) \end{aligned}$$

where we defined the time-varying, fractional- modulated window,

$$\mathrm{p}_i(n,\ell) = \sum_{m=0}^{M-1} \tilde{g}_i(n-mL)\tilde{\gamma}_i^*(\ell-mL)e^{j\frac{1}{2}(\ell^2-n^2))\cot\alpha}$$

The equation in (13) can be interpreted as the average of short-time Fourier transforms with scaled, time-dependent and non-sinusoidal modulated windows $p_i(n,\ell)$. The fractional evolutionary spectrum is then obtained as before. Furthermore, $\mathrm{p}_i(n,\ell)$ can be calculated independent of the signal and then the calculation of the ES can be achieved very efficiently using FFT. In Fig. 2, we show an example of this time-varying fractional window $p_i(n,\ell)$ for $\alpha = \pi/4$ at time instants $n = 32$ and $n = 80$ and their spectrogram together.

5. EXPERIMENTAL RESULTS

We consider a signal composed of two crossing chirps -one with increasing frequency and the other decreasing frequency- with angles $\pi/4$ and $-\pi/4$ is considered. This signal is first analyzed with $\alpha = \pi/4, L = 4, K = 128$, and $I = 4$ and ES estimate is given in Fig. 3. Notice that increasing chirp is represented with higher TF localization since the fraction order matches this component. Then the ES is estimated using $\alpha = -\pi/4$ fraction order and same sampling parameters and the result is given in Fig. 4. As shown from the figures, the component that is matched by the fraction order is displayed in the TF plane with a high resolution.

6. CONCLUSIONS

In this paper, we present a method for fractional evolutionary spectral analysis of discrete-time, non- stationary signals. The evolutionary kernel is obtained via the coefficients of a fractional Gabor expansion. We give the completeness and biorthogonality conditions of this new expansion. Simulations show that the fractional method gives high resolution ES results if the analysis fraction order match the frequency content of the signal. Hence, for an arbitrary signal, the fraction order α can be chosen from a set of values $\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$ by maximizing a concentration criteria similar to the method used in [5, 7].

7. REFERENCES

[1] Cohen, L., *Time-Frequency Analysis*. Prentice Hall, Englewood Cliffs, NJ, 1995.

[2] Priestley, M.B., "Evolutionary Spectra and Non–stationary Processes," *J. of Royal Statistical Society, B*, Vol.27, No. 2, pp. 204–237, 1965.

[3] Akan, A., and Chaparro, L.F., "Multi–window Gabor Expansion for Evolutionary Spectral Analysis," *Signal Processing*, Vol. 63, pp. 249–262, Dec. 1997.

[4] Wexler, J., and Raz, S., "Discrete Gabor Expansions," *Signal Processing*, Vol. 21, No. 3, pp. 207–220, Nov. 1990.

[5] Jones, D.L., and Parks, T.W., "A High Resolution Data–Adaptive Time–Frequency Representation," *IEEE Trans. on Signal Proc.*, Vol. 38, No. 12, pp. 2127–2135, Dec. 1990.

[6] Mallat, S., and, Zhang, Z., "Matching Pursuit with Time–Frequency Dictionaries," *IEEE Trans. on Signal Proc.*, Vol. 41, No. 12, pp. 3397–3415, Dec. 1993.

[7] Jones, D.L., and Baraniuk, R.G., "A Simple Scheme For Adapting Time–Frequency Representations," *IEEE Trans. on Signal Proc.*, Vol. 42, No. 12, pp. 3530–3535, Dec. 1994.

[8] Bultan A., "A Four–Parameter Atomic Decomposition of Chirplets," *IEEE Tans. on Signal Proc.*, Vol. 47 pp. 731–745, 1999.

[9] Bastiaans, M.J., and van Leest, A.J., "From the Rectangular to the Quincunx Gabor Lattice via Fractional Fourier Trasformation," *IEEE Signal Proc. Letters*, Vol. 5, No. 8, pp. 203–205, 1998.

[10] Pei, S.C., Yeh, M.H., and Luo, T.L., "Fractional Fourier Series Expansion for Finite Signals and Dual Extension to Discrete–Time Fractional Fourier Transform," *IEEE Trans. on Signal Proc.*, Vol. 47, No. 10, pp. 2883–2888, Oct. 1999.

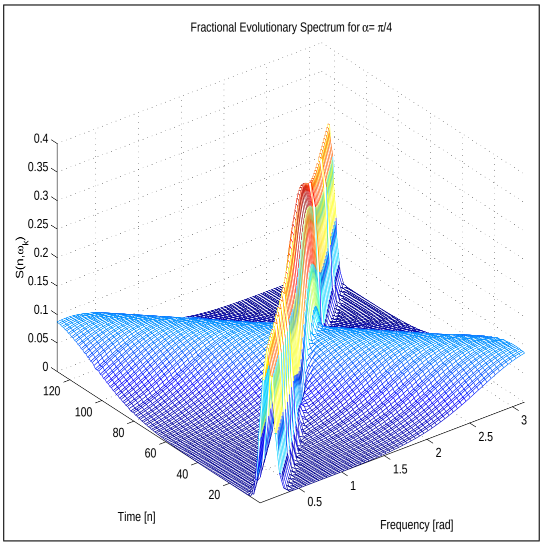

Fig. 3. ES estimate of the crossing chirp signal with $\alpha = \pi/4$.

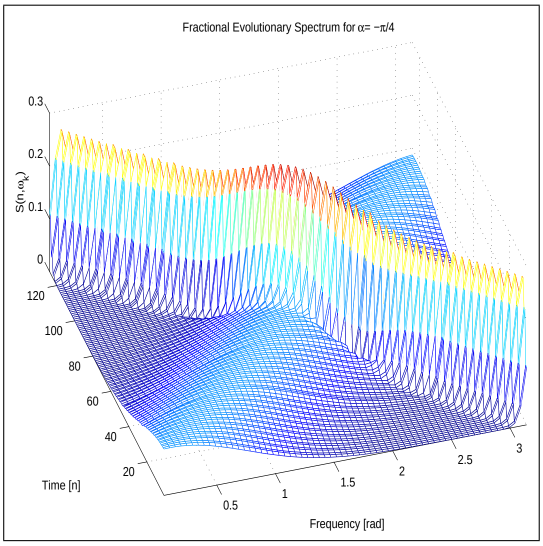

Fig. 4. ES estimate using $\alpha = -\pi/4$.