TWO APPROACHES FOR THE ESTIMATION OF TIME-VARYING AMPLITUDE MULTICHIRP SIGNALS

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ABSTRACT

This paper addresses the problem of time-varying amplitude multichirp signals parameters estimation. We compare two approaches which require a model for the amplitude. First, we use a basis of time-localized functions associated with Bayesian estimation. Secondly, we use an autoregressive model associated with a mixed high-order ambiguity function/Kalman filter estimation. Results show that both methods are efficient to solve this estimation problem.

1. INTRODUCTION

Chirps are one of the most studied class of signals, as they appear in numerous applications such as sonar, radar, acoustics, etc. However, authors generally consider monochirp signals (i.e., signals with one chirp component at a given time), and the chirps amplitude is almost always assumed stationary.

In this paper, we address the general problem of multichirp signals, each chirp component having a time-varying amplitude. Our model is more realistic than the previous simple cases. More precisely, the signal model considered here is, for $n = 1, \dots, N$:

$$x[n] = \sum_{k=1}^{K} a_k[n] \cos(\varphi_k + 2\pi f_k n + \pi s_k n^2) + w[n] \qquad (1)$$

where n is the discrete time index, K is the total number of chirps, $\{a_k[n]\}_{1,\dots,K}$, are time-varying amplitudes, $\{\varphi_k\}_{1,\dots,K}$ are initial phases, $\{f_k\}_{1,\dots,K}$ are initial frequencies and $\{s_k\}_{1,\dots,K}$ are chirp slopes. The additive noise w is assumed zero-mean white Gaussian with variance σ^2. In particular, an example application in which such signals arise can be found in [1] (loudspeakers fault detection).

The set of parameters defining this model is large, and practical estimation requires a time-varying amplitudes $\{a_k[n]\}_{1,\dots,K}$ model to be defined. We address two different models, each associated with a dedicated algorithm. The first model consists of projecting $\{a_k[n]\}_{1,\dots,K}$ on a basis of time-localized functions, and Bayes estimation is performed using MCMC. The second model relies on an autoregressive modelling of amplitudes time evolution, and is implemented with a mixed high-order ambiguity function estimator / Kalman filter.

This paper is organised as follows. In Section 2, the model relying on a set of basis functions (referred to as *projected amplitudes model*) is described, and the estimation algorithm is summarised. In Section 3, the model relying on autoregressive amplitudes modelling (referred to as *autoregressive amplitudes model*) is presented and the estimation algorithm is outlined. Section 4 provides simulation results and compares the two models. We conclude about the good qualities and drawbacks of both methods.

2. PROJECTED AMPLITUDES MODEL

In this section, we present a model in which the time-varying amplitudes are projected on a set of $L + 1$ basis functions, denoted $\phi_l[n]$, $n = 1, \dots, N$:

$$a_k[n] \;=\; \sum_{l=0}^{L} a_k^l \phi_l[n] \qquad (2)$$

where a_k^l denotes the amplitude coefficient corresponding to basis function $\phi_l[n]$. For a given l, $\phi_l[n]$ is obtained by translating a mother basis function $\phi[n]$ so that $\phi_l[n]$ is centered around time $l\Delta_n$ with $\Delta_n = \frac{N-1}{L}$. More precisely,

$$\phi_l[n] \;=\; \phi[n - l\Delta_n] \qquad (3)$$

In practice, $\phi[n]$ is selected as a smooth, non-oscillating unitary energy function such as a cubic spline, a Hanning window, etc. Using Eq.'s (1)-(2), one has

$$\begin{aligned} x[n] \;=\; & \sum_{k=1}^{K}\sum_{l=0}^{L} \big[a_{c,k}^l \cos\left(2\pi\, f_k\, n + \pi\, s_k\, n^2\right) \\ & + a_{s,k}^l \sin\left(2\pi\, f_k\, n + \pi\, s_k\, n^2\right)\big]\phi_l[n] + w[n] \end{aligned} \qquad (4)$$

with $a_{c,k}^l = a_k^l \cos(\varphi_k)$ et $a_{s,k}^l = -a_k^l \sin(\varphi_k)$ (this new parameterisation is useful for MCMC simulations, and is often adopted in similar contexts [2, 3]). Decomposition of time-varying amplitudes on a set of basis functions is often a good solution within a Bayesian framework, see [3] for another application. In vector-matrix form, Eq. (4) becomes:

$$\mathbf{X} = \mathbf{D}\,\mathbf{a} + \mathbf{W} \qquad (5)$$

where $\mathbf{X} = \left[x[1], \dots, x[n]\right]^T$ is the signal vector. Similarly, $\mathbf{W}$ denotes the noise vector and $\mathbf{a}$ is the vector obtained by stacking the amplitudes, and $\mathbf{D}$ is a matrix whose columns contain modulated basis functions (terms $\phi_l[n]\cos(2\pi\, f_k\, n + \pi\, s_k\, n^2)$ and $\phi_l[n]\sin(2\pi\, f_k\, n + \pi\, s_k\, n^2)$ for all $n = 1, \dots, N$), see, e.g.,

[2] for a similar setting. The likelihood of $\mathbf{X}$ is:

$$p(\mathbf{X}|\mathbf{a},\mathbf{s},\mathbf{f},\sigma^2) = (2\pi\sigma^2)^{\frac{-N}{2}} \exp[-\frac{(\mathbf{X}-\mathbf{D}\,\mathbf{a})^{\mathrm{t}}(\mathbf{X}-\mathbf{D}\,\mathbf{a})}{2\sigma^2}] \tag{6}$$

where $\mathbf{s} = [s_1, \ldots, s_K]^T$ and $\mathbf{f} = [f_1, \ldots, f_K]^T$.

We now endow this model with Bayesian assumptions. The posterior distribution of the multichirp signal parameters is given by

$$p(\mathbf{a},\mathbf{s},\mathbf{f},\sigma^2|\mathbf{X}) \propto p(\mathbf{X}|\mathbf{a},\mathbf{s},\mathbf{f},\sigma^2)p(\mathbf{a},\mathbf{s},\mathbf{f},\sigma^2), \tag{7}$$

where we assume the following hierarchical structure for the prior distribution

$$p(\mathbf{a},\mathbf{s},\mathbf{f},\sigma^2) \propto p(\mathbf{a}|\mathbf{s},\mathbf{f},\sigma^2)p(\mathbf{s})p(\mathbf{f})p(\sigma^2). \tag{8}$$

Here, we propose to select vague priors but other choices could be implemented, depending on the amount of prior information available. Frequencies (resp. slope parameters) are assumed independent, uniformly distributed between 0 and 0.5, i.e. $\mathbf{f} \sim \mathcal{U}[0, 0.5]$ (resp. $\mathbf{s} \sim \mathcal{U}[0, 0.5/N]$[1]). The noise variance is distributed according to a inverse gamma distribution (conjugate prior) $\sigma^2 \sim \mathcal{IG}(\alpha,\beta)$. The amplitude parameter $\mathbf{a}$ is zero-mean Gaussian with covariance matrix $\sigma^2\Sigma_a$. Note that the covariance matrix is scaled by the noise variance. An efficient choice for Σ_a^{-1} is $\frac{1}{\delta^2}D^TD$, known as the g-prior distribution [4, 2]. Standard calculations lead to:

$$\begin{aligned} p(\mathbf{f},\mathbf{s}|\mathbf{X}) \quad \propto \quad & \prod_{k=1}^{K} \mathbb{I}_{[0,0.5]}(f_k)\mathbb{I}_{[0,\frac{0.5}{N}]}(s_k)\frac{1}{(\delta^2+1)^{K(L+1)}} \\ & \times \left[\mathbf{X}^T\left(\mathbf{I}_N - \mathbf{D}\mathbf{P}^T\mathbf{D}^T\right)\mathbf{X} + 2\beta\right]^{-\alpha-\frac{N}{2}} \end{aligned} \tag{9}$$

with $\mathbf{P}^{-1} = \mathbf{D}^T\mathbf{D} + \Sigma_a^{-1}$ and $\mathbb{I}_{[u,v]}(x) = 1$ if $u \leq x \leq v$ and 0 otherwise. $\mathbf{I}_N$ is the identity matrix of size N. Moreover,

$$p(\sigma^2|\mathbf{f},\mathbf{s}) = \mathcal{IG}\left(\alpha + \frac{N}{2}, \frac{\mathbf{X}^T\left(\mathbf{I}_N - \mathbf{D}\mathbf{P}^T\mathbf{D}^T\right)\mathbf{X}}{2} + \beta\right) \tag{10}$$

$$p(\mathbf{a}|\mathbf{f},\mathbf{s},\sigma^2) = \mathcal{N}\left(\mu\,,\,\sigma^2\,\mathbf{P}\right) \text{ with } \mu \;=\; \mathbf{P}\,\mathbf{D}^T\,\mathbf{X} \tag{11}$$

The three hyperparameters α, β and δ^2 are selected such that $\alpha = \beta = 0$ and $\delta^2 \sim \mathcal{IG}(\alpha_\delta, \beta_\delta)$. The estimation of the chirps parameters $\mathbf{a}$, $\mathbf{f}$, $\mathbf{s}$, σ^2 is done through the MMSE estimate, e.g., for the amplitudes

$$\widehat{\mathbf{a}} \;=\; \int \mathbf{a}\,p(\mathbf{a},\mathbf{f},\mathbf{s},\sigma^2|\mathbf{X})\,d\mathbf{a}\,d\mathbf{f}\,d\mathbf{s}\,d\sigma^2 \tag{12}$$

which is well approached by the Monte Carlo approximation

$$\widehat{\mathbf{a}} \;\approx\; \frac{1}{M}\sum_{m=1}^{M}\widetilde{\mathbf{a}}^{(m)} \tag{13}$$

This requires to have samples $\{\mathbf{a}^{(m)}, \mathbf{f}^{(m)}, \mathbf{s}^{(m)}, \sigma^{2\,(m)}\}$, $m = 1, \ldots, M$ distributed according to $p(\mathbf{a},\mathbf{f},\mathbf{s},\sigma^2)$. The algorithm aimed at generating such samples (MCMC algorithm) is very similar to the well described sampler in [2]. For the sake of brevity, we have not presented here the case where the number K of chirps is unknown, but provided a prior on K is defined, its estimation is straightforward (using reversible jump MCMC).

[1]Here, we assume positive slopes, but this choice is indicative, and extensions to other minimum/maximum bounds are still possible.

3. AUTOREGRESSIVE AMPLITUDES MODEL

The second time-varying chirps amplitudes model relies on a first order autoregressive modelling of the amplitudes, namely, for all $k = 1, \ldots, K$:

$$a_k[n+1] \;=\; a_k[n] + v[n] \tag{14}$$

We adopt here a completely different estimation approach. No Bayes assumptions are taken, and we rather rely on an estimation method specially designed for polynomial phase signals (chirps are order 2 polynomial phase signals). We first assume that the signals are monocomponent, i.e., signals are made of only one chirp (this assumption will be relaxed later in this paper). The method we start from is based on the high-order ambiguity function (HAF) introduced by Peleg [5]. In order to estimate the amplitudes of each component, we associate it to a Kalman filter.

The high-order ambiguity function of a complex-valued signal x is the Fourier Transform (FT) of the operator $\mathrm{D}\mathcal{P}_H$, see [5] for a complete definition. The main interest of this function is summarised in the following property.

Let x be a complex-valued polynomial phase signal such that $x[n] = \exp\left[j2\pi\sum_{h=0}^{H}\nu_h n^h\right]$, then:

$$\mathrm{D}\mathcal{P}_H(x)[n,l] \;=\; \exp\left[j2\pi(\phi_0(l)n + \phi_1(l))\right] \tag{15}$$

where n is time and l is time-lag, and

$$\phi_0(l) = H!\,l^{H-1}\nu_H \tag{16}$$

$$\phi_1(l) = (H-1)!\,l^{H-1}\nu_{H-1} - 0.5(H-1)H!\,l^H\nu_H \tag{17}$$

and its FT is the order H ambiguity function of x.

In practice, this property provides an efficient way to estimate the highest order phase coefficient $\hat{\nu}_H$, because a peak appears in the HAF at the frequency $\hat{\nu}_H$. In order to estimate lower order coefficients $\nu_0, \ldots, \nu_{H-1}$, the following algorithm can be implemented:

Algorithm 1

1. Initialisation
- Set $h \leftarrow H$ and set $y[n] \leftarrow x[n]$
2. Polynomial phase coefficients estimation
- while $h \geq 0$, do
  - Choose l
  - Estimate $\hat{\nu}_h$ by computing the FT of $\mathrm{D}\mathcal{P}_h(y)[n,l]$
  - Demodulate $y[n]$ with $\exp(-j\hat{\nu}_h n^h)$, i.e. set $y[n] \leftarrow y[n]\exp(-j\hat{\nu}_h\,n^h)$
  - set $h \leftarrow h-1$

In presence of additive noise, the estimation is accurate up to a moderate SNR [5], and remains computationally cheap. Moreover, this estimation technique is robust to slowly time-varying amplitudes.

Once the frequency and slope parameters are known, a Kalman filter is implemented so as to estimate the time-varying amplitudes, see [6] for more details. Note that the initial phase can be estimated at this step by considering complex valued amplitudes $\theta_k[n] = a_k[n] \exp[j\varphi_k]$. In Algorithm 1, the last iteration provides the initial frequency parameter, and the remaining demodulated signal is an estimate $\hat{\theta}_k[n]$ of $\theta_k[n]$. One could simply consider $\hat{\theta}_k[n]$ as a good amplitudes estimate, but we can refine it by implementing the Kalman filter on the model:

$$\theta_k[n+1] = \theta_k[n] + v[n] \quad (18)$$

$$x[n] = \left[\exp[j(2\pi \hat{f}_k n + \pi \hat{s}_k n^2)\right] \theta_k[n] + w[n] \quad (19)$$

where v and w are zero-mean white Gaussian noises of variances q and σ^2. Eq. (18) is the evolution equation whereas Eq. (19) is the observation equation[2]. In practice, a forward-backward-forward filter is implemented.

We now address the case of multichirp signals. If $x[n]$ contains multiple components of order H, its HAF has several peaks: some have high energy and correspond to real chirp components and others have small energy, and correspond to cross terms. Peaks are considered one at a time in a hierarchical order, from the highest energy peak to the lowest energy peak. For each peak, Algorithm 1 is implemented as if there were only one chirp. Once parameters s_k and f_k are estimated for this chirp, the previous forward-backward-forward Kalman filter is implemented so as to estimate the time-varying chirp amplitude. Then, the fully estimated chirp component is subtracted from $x[n]$, and the procedure is iterated. It ends whenever no HAF peak has a significant energy. This procedure, introduced in [5], is accurate for polynomial phase coefficients estimation.

Similar to the monochirps case, a second Kalman filtering step is now implemented in order to refine amplitudes estimates. In matrix-form, the full model is:

$$\boldsymbol{\theta}[n+1] = \boldsymbol{\theta} + \mathbf{V}[n] \quad (20)$$

$$x[n] = C(\hat{\boldsymbol{s}}, \hat{\boldsymbol{f}})[n]\boldsymbol{\theta}[n] + w[n] \quad (21)$$

where $\boldsymbol{\theta}[n] = \left[\theta_1[n], \ldots, \theta_K[n]\right]$. The covariance matrix of $\mathbf{V}$ is $q\mathbf{I}_K$ and w has variance σ^2.

4. SIMULATION RESULTS

In this section, we apply the two previous methods to the estimation of a multichirp signal embedded in a Gaussian white noise. We select $K = 4$ chirps, whose phase parameters are listed in table 1. The time-varying amplitudes are generated with a Wiener smoothed filter with $N = 512$, and various SNRs (i.e., various σ^2) are tested.

For the projected amplitudes model, we choose $L + 1 = 8$ basis functions, and 10000 MCMC runs are performed. For the autoregressive amplitudes model, the HAF is only computed for time-lag $l = N/H$ which is shown to be optimal [5]. The variance q for the Kalman filter is $q = 10^{-5}$.

In Fig. 1, the amplitudes estimation results for both models and each chirp component are displayed for SNR = 20dB. Both methods yield accurate estimates. However, estimates obtained with the autoregressive amplitudes model are more accurate than

Initial phases	Initial frequencies	Slopes
0	0.00090703	5.9688 10^{-5}
$\pi/2$	0.031467	1.1871 10^{-4}
π	0.092246	2.3608 10^{-4}
$2\pi/3$	0.21312	4.6952 10^{-4}

Table 1. Phase parameters of the four components of the test signals.

estimates obtained with the projected amplitudes model. Moreover, the Bayesian computations (MCMC) are more difficult to handle than the high-order ambiguity function/Kalman filter based approach.

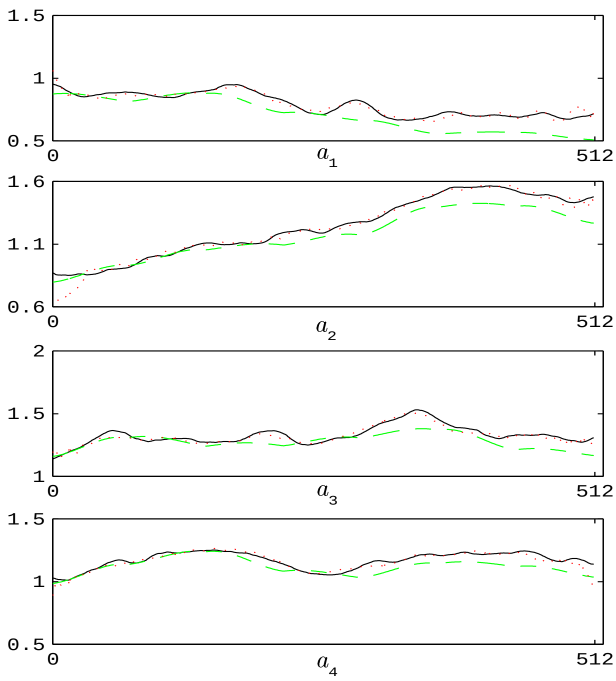

Fig. 1. Estimation of the time-varying amplitudes of the four components with the projected amplitudes model (dashed lines) and the autoregressive model (dotted lines). The true amplitudes are also plotted for comparison (solid lines).

We now apply both estimation algorithms to 50 signals with the phase parameters given in Tab. 1 and three different SNRs (20, 10 and 0dB)[3]. The phase parameters estimates are listed in Tab. 2 together with standard deviations. For the amplitudes, we provide the mean-square deviation between the true time-varying amplitude and the estimated amplitude.

Interestingly, both methods provide accurate estimates of the slope and of the initial frequency, and poor estimates of the initial phase. The lower the SNR, the lower the accuracy of both methods (as expected). However, the Bayesian model yields better slopes and initial frequency estimates around SNR=0dB. Nevertheless, the estimation accuracy of the amplitude is always better with the

[2] In practice, σ^2 is tuned so as to yield satisfactory results.

[3] The 50 signals differ from the noise realization.

	Bayesian			HAF		
	20	10	0	20	10	0
Slopes ($\times 10^{-4}$)						
s_1	0.586 (0.0119)	0.580 (0.0238)	0.506 (0.0397)	0.585 (0.0010)	0.586 (0.0027)	0.669 (0.4103)
s_2	1.188 (0.0087)	1.193 (0.0195)	1.188 (0.0470)	1.186 (0.0068)	1.176 (0.0112)	1.160 (0.3362)
s_3	2.366 (0.0064)	2.367 (0.0178)	2.363 (0.0483)	2.341 (0.0035)	2.344 (0.0114)	2.245 (0.8142)
s_4	4.704 (0.0098)	4.699 (0.0209)	4.703 (0.0343)	4.700 (0.0052)	4.705 (0.0075)	4.703 (0.0139)
Initial Frequencies($\times 10^{-1}$)						
f_1	0.013 (0.0040)	0.016 (0.0062)	0.0271 (0.0097)	0.013 (0.0002)	0.013 (0.0006)	0.005 (0.0405)
f_2	0.315 (0.0027)	0.314 (0.0068)	0.3146 (0.0147)	0.317 (0.0016)	0.32 (0.0027)	0.315 (0.0873)
f_3	0.922 (0.0019)	0.922 (0.0051)	0.9239 (0.0120)	0.932 (0.0010)	0.931 (0.0031)	0.932 (0.0468)
f_4	2.131 (0.0026)	2.133 (0.0052)	2.1314 (0.0107)	2.132 (0.0018)	2.131 (0.0026)	2.131 (0.0039)
Initial Phases						
φ_1	-0.06 (0.9125)	-0.199 (0.8776)	-0.541 (0.8343)	-0.299 (0.0116)	-0.298 (0.0323)	-0.173 (1.4205)
φ_2	1.7 (1.1504)	1.811 (1.0730)	1.658 (0.8737)	1.21 (0.0724)	1.094 (0.1329)	1.328 (1.4807)
φ_3	3.27 (0.9084)	2.942 (0.8776)	2.601 (0.8343)	2.2 (0.0599)	2.421 (0.8567)	3.799 (2.5399)
φ_4	2.516 (0.6847)	2.503 (0.6920)	2.412 (0.7434)	2.028 (0.1509)	2.171 (0.2139)	2.096 (0.3178)
Amplitudes						
a_1	2.1531	1.5419	1.0047	$0.3529\ 10^{-3}$	0.0014	0.0159
a_2	1.6531	1.8579	1.7390	$0.2550\ 10^{-3}$	0.0013	0.0241
a_3	2.2222	1.9460	1.9555	$0.2851\ 10^{-3}$	0.0016	0.0284
a_4	1.3455	1.4933	1.4555	$0.2009\ 10^{-3}$	0.0014	0.0151

Table 2. Mean of the phase parameters estimation and, in brackets, their standard deviation. Mean-square deviation between the true time-varying amplitude and the estimated amplitude. It focus on 50 signals for each signal-to-noise ratio (20, 10 and 0dB).

HAF/Kalman filter based method.

Finally, we note that the computational cost is dramatically higher for the Bayesian model, and computations were 2093 times longer with MCMC than with HAF/Kalman filter.

5. CONCLUSION

In this paper, we have developed two methods for the estimation of the chirp phase parameters of multichirp signals as well as their time-varying amplitudes. Simulation results show that Bayesian inference is accurate for the estimation of the phase parameters. Nevertheless, at high SNR (20 dB), the HAF/Kalman filter based method also yields accurate estimates. Furthermore, this latter method is also more accurate for amplitudes estimation, whatever the SNR. Note that our estimation methods can easily be extended to higher-order polynomial phase signals. Further investigations have shown that these time-varying amplitudes estimates could be efficiently used for signal classification (in the problem of loudspeaker fault detection [1]). This extends the work in [7] where chirps were classified from their phase parameters.

6. REFERENCES

[1] M. Davy and C. Doncarli, “A New Nonstationary Test Procedure for Improved Loudspeaker Fault Detection,” *AES Journal*, vol. 50, no. 6, June 2002.

[2] C. Andrieu and A. Doucet, “Joint bayesian model selection and estimation of noisy sinusoids via reversible jump mcmc,” *IEEE Transactions on Signal Processing*, vol. 47, no. 10, pp. 2667–2676, October 1999.

[3] M. Davy and S. Godsill, “Bayesian harmonic models for musical signal analysis,” in *Seventh Valencia International meeting (Bayesian Statistics 7)*, Oxford University Press, Ed., 2002, to appear.

[4] A. Zellner, “On assessing prior distributions and bayesian regression analysis with g-prior distributions,” *Bayesian Inference and Decision Techniques*, 1986.

[5] S. Peleg, *Estimation and detection with discrete polynomial transform*, Ph.D. thesis, University of California, Davis, 1993.

[6] H. Cottereau, J.M. Piasco, and C. Doncarli, “Identification de signaux multicomposantes à phase polynomiale et à amplitude variable,” in *18e colloque GRETSI sur le Traitement du Signal et des Images*, Toulouse, France, 10-13 septembre 2001.

[7] M. Davy, C. Doncarli, and J.-Y. Tourneret, “Classification of chirp signals using hierarchical bayesian learning and mcmc methods,” *IEEE Transactions on Signal Processing*, vol. 50, no. 2, pp. 377 –388, February 2002.