ROBUSTON METHODS FOR STABLE STATISTICAL SIGNAL PROCESSING: PRINCIPLES AND APPLICATION TO NONSTATIONARY SIGNAL ESTIMATION

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ABSTRACT

We introduce a reduced-detail paradigm for nonstationary statistical signal processing with enhanced performance. Time-frequency localized subspace signal components (called *robustons*) are used as atomic entities for statistical signal modeling and processing. Robuston signal processing employs special time-varying filters that allow an efficient on-line implementation, and statistical signal descriptors that can be estimated in a stable manner by means of *intra-subspace averaging*. We develop the principles of robuston signal processing and consider optimal nonstationary signal estimation as a specific application. The performance advantages of the resulting "robuston Wiener filters" are assessed by means of simulations.

1. INTRODUCTION

A statistical signal model or a method for statistical signal processing is of little practical value if it is so detailed that the parameters involved cannot be estimated with sufficient accuracy. This is especially true in nonstationary environments where averaging over longer time periods cannot be used.

In this paper, therefore, we propose a reduced-detail paradigm for nonstationary statistical signal processing with improved statistical stability. Signals are decomposed into time-frequency localized subspace components (termed *robustons*), and *each robuston is considered as an atomic entity for statistical signal modeling and processing*. Also, not all statistical dependencies between different robustons are taken into account. The relevant second-order statistics can be estimated by means of *intra-subspace averaging*, which results in signal processing methods with enhanced statistical robustness. In fact, this *robuston paradigm* generalizes a previously proposed scheme that is robust in a minimax sense [1, 2].

The paper is organized as follows. After a review of the robuston subspace decomposition [1, 2] in this section, Section 2 proposes *robuston filters* that are the workhorse of robuston signal processing. The statistical signal descriptors (*robuston correlations*) used and their stable estimation are discussed in Section 3. Section 4 presents a corresponding statistical signal model (*robuston processes*). Finally, Section 5 develops the application of robuston signal processing to optimal nonstationary signal estimation (Wiener filtering) and assesses the performance advantages of robuston Wiener filters.

Robuston decomposition. We use a decomposition of a discrete-time signal $x[n]$ into subspace components (*robustons*) $x_{k,l}[n]$,

$$x[n] = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{L-1} x_{k,l}[n], \qquad (1)$$

where k is a time index and l is a frequency index [1, 2]. Each robuston $x_{k,l}[n]$ lies in a P-dimensional linear signal subspace $\mathscr{X}_{k,l} =$

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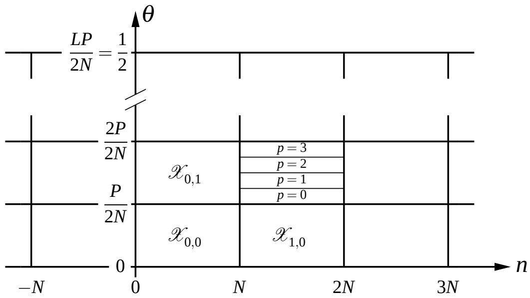

Figure 1: *Time-frequency localization of the signal subspaces $\mathscr{X}_{k,l}$, with $\mathscr{X}_{1,1}$ blown up into its LCB functions (subspace dimension $P = 4$ assumed). Only positive frequencies are shown.*

$\text{span}\{u_{k,l}^{(p)}[n]\}_{p=0,\cdots,P-1}$ that is spanned by the following P local cosine basis (LCB) functions [3, 4]:

$$u_{k,l}^{(p)}[n] = \sqrt{\frac{2}{N}} \cos\left(\pi \frac{lP+p+1/2}{N}(n-kN)\right) w[n-kN],$$

with $k \in \mathbb{Z}$, $l \in [0, L-1]$, and $p \in [0, P-1]$. Here, $N = LP$ is the block length and $w[n]$ is a suitably chosen window with effective time duration N [3, 4]. As shown in Fig. 1, the robuston decomposition corresponds to a uniform tiling of the time-frequency plane. Specifically, $\mathscr{X}_{k,l}$—and, thus, the robuston $x_{k,l}[n] \in \mathscr{X}_{k,l}$—is effectively localized within the time interval (block) $[kN, (k+1)N]$ and the frequency band $\left[\frac{lP}{2N}, \frac{(l+1)P}{2N}\right]$. The LCB functions $u_{k,l}^{(p)}[n]$ form an orthonormal basis of $l^2(\mathbb{Z})$, and thus $\{\mathscr{X}_{k,l}\}_{k \in \mathbb{Z}, l \in [0,L-1]}$ is an orthogonal partition of $l^2(\mathbb{Z})$. The robustons can be calculated as

$$x_{k,l}[n] = (\mathbf{P}_{k,l}x)[n] = \sum_{p=0}^{P-1} \alpha_{k,l}^{(p)} u_{k,l}^{(p)}[n], \qquad (2)$$

where

$$\alpha_{k,l}^{(p)} = \langle x, u_{k,l}^{(p)} \rangle = \sum_{n=-\infty}^{\infty} x[n] u_{k,l}^{(p)*}[n],$$

and $\mathbf{P}_{k,l}$ denotes the orthogonal projection operator on $\mathscr{X}_{k,l}$. We note that $u_{k+m,l}^{(p)}[n] = u_{k,l}^{(p)}[n-mN]$ and hence $x_{k,l}[n-mN] \in \mathscr{X}_{k+m,l}$.

2. ROBUSTON FILTERS

Definition and expressions. *Robuston filters* (RFs) are the workhorse of robuston signal processing. An RF $\mathbf{H}$ is a linear, time-varying filter that relates the robustons $x_{k,l}[n]$ of the filter input $x[n]$ and the robustons $y_{k,l}[n]$ of the filter output $y[n] = (\mathbf{H}x)[n]$ as

$$y_{k,l}[n] = \sum_{k'=k-M_1}^{k+M_2} h_{k,k';l}\, x_{k',l}[n-(k-k')N]. \qquad (3)$$

That is, $y_{k,l}[n]$ is a weighted sum of the (suitably time-shifted) $x_{k',l}[n]$ located in the *same* (lth) frequency band—thus, there is no "cross talk" between different frequency bands—and within a local neighborhood $k' \in [k-M_1, k+M_2]$ of the current (kth) block. The filter coefficient $h_{k,k';l}$ describes the mapping of $x_{k',l}[n]$ to $y_{k,l}[n]$. The filter length is given by M_1+M_2+1. Note that an RF processes each robuston as an entity, without distinguishing between the individual LCB components within the robuston.

Using $y[n] = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{L-1} y_{k,l}[n]$ and $x_{k,l}[n] = (\mathbf{P}_{k,l} x)[n]$, the input-output relation of the RF $\mathbf{H}$ defined by (3) is obtained as

$$(\mathbf{H}x)[n] = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{L-1} \left[\sum_{k'=k-M_1}^{k+M_2} h_{k,k';l} \, (\mathbf{P}_{k',l} x)[n-(k-k')N] \right].$$

Introducing the block time-shift operator $\mathbf{S}_m$ as $(\mathbf{S}_m x)[n] = x[n-mN]$, we can write the RF operator as

$$\mathbf{H} = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{L-1} \mathbf{H}_{k,l}, \quad \text{with } \mathbf{H}_{k,l} = \sum_{k'=k-M_1}^{k+M_2} h_{k,k';l} \, \mathbf{S}_{k-k'} \mathbf{P}_{k',l}. \quad (4)$$

The component system $\mathbf{H}_{k,l}$ is associated to $\mathscr{X}_{k,l}$ in that $(\mathbf{H}_{k,l} x)[n] = y_{k,l}[n] \in \mathscr{X}_{k,l}$ (cf. (3)). Because of (2) and $u_{k+m,l}^{(p)}[n] = u_{k,l}^{(p)}[n-mN]$, the elementary systems $\mathbf{S}_{k-k'} \mathbf{P}_{k',l}$ can be expressed as

$$(\mathbf{S}_{k-k'} \mathbf{P}_{k',l} x)[n] = \sum_{p=0}^{P-1} \alpha_{k',l}^{(p)} u_{k,l}^{(p)}[n], \quad \text{with } \alpha_{k,l}^{(p)} = \langle x, u_{k,l}^{(p)} \rangle. \quad (5)$$

For $M_1 = M_2 = 0$, we obtain $\mathbf{H}_{k,l} = h_{k,k;l} \mathbf{P}_{k,l}$, and thus the RF reduces to the weighted sum of projectors [1, 2]

$$\mathbf{H} = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{L-1} h_{k,k;l} \mathbf{P}_{k,l} \quad \text{for } M_1 = M_2 = 0.$$

Implementation and complexity. RFs allow an efficient on-line implementation. For the kth block of length N, this implementation consists of the following three steps:

1. *LCB analysis*: Calculation of $\alpha_{k,l}^{(p)} = \langle x, u_{k,l}^{(p)} \rangle$.
2. "*Subband filtering*": Calculation of $\tilde{\alpha}_{k,l}^{(p)} = \sum_{k'=k-M_1}^{k+M_2} h_{k,k';l} \, \alpha_{k',l}^{(p)}$.
3. *LCB synthesis*: Calculation of $y_k[n] = \sum_{l=0}^{L-1} \sum_{p=0}^{P-1} \tilde{\alpha}_{k,l}^{(p)} \, u_{k,l}^{(p)}[n]$.

Using efficient discrete cosine transform algorithms for the LCB analysis and synthesis steps [3, 4], the computational complexity of this implementation is $\mathscr{O}\big(N(2\log_2 N + M_1 + M_2)\big)$ per signal block and thus $\mathscr{O}(2\log_2 N + M_1 + M_2)$ per output signal sample. Another practically attractive feature of RFs is that they allow easy control of the physically important parameters time and frequency. This is due to the time-frequency localization of the subspaces $\mathscr{X}_{k,l}$ (cf. Fig. 1).

Properties. Some theoretical properties of RFs are summarized in the following. For notational convenience, we will use the *coefficient matrices* $\underline{\mathrm{H}}_l$ defined as $[\underline{\mathrm{H}}_l]_{k,k'} = h_{k,k';l}$.

P_1: The *identity operator* $\mathbf{I}$ is an RF with coefficients $h_{k,k';l} = \delta_{k,k'}$ or equivalently $\underline{\mathrm{H}}_l = \underline{\mathrm{I}}$ (the identity matrix) for all l.

P_2: The *adjoint* $\mathbf{H}^+$ of an RF $\mathbf{H}$ is an RF with coefficients $h_{k',k;l}^*$ or $\underline{\mathrm{H}}_l^H$ (^{H} denotes Hermitian matrix transpose).

P_3: A *weighted sum* (weighted parallel connection) $\mathbf{H} = a\mathbf{H}^{(1)} + b\mathbf{H}^{(2)}$ of two RFs $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ is an RF with coefficients $h_{k,k';l} = a h_{k,k';l}^{(1)} + b h_{k,k';l}^{(2)}$ or $\underline{\mathrm{H}}_l = a\underline{\mathrm{H}}_l^{(1)} + b\underline{\mathrm{H}}_l^{(2)}$.

P_4: The *composition* (series connection) $\mathbf{H} = \mathbf{H}^{(2)} \mathbf{H}^{(1)}$ of two RFs is an RF with $h_{k,k';l} = \sum_\kappa h_{k,\kappa;l}^{(2)} h_{\kappa,k';l}^{(1)}$ or $\underline{\mathrm{H}}_l = \underline{\mathrm{H}}_l^{(2)} \underline{\mathrm{H}}_l^{(1)}$.

P_5: If the *inverse* $\mathbf{H}^{-1}$ of an RF $\mathbf{H}$ exists, it is an RF with coefficient matrices $\underline{\mathrm{H}}_l^{-1}$.

3. ROBUSTON CORRELATIONS

Definition. Robuston processing is based on a reduced-detail description of the second-order signal statistics. For a zero-mean, nonstationary random process $x[n]$, the LCB expansion coefficients $\alpha_{k,l}^{(p)} = \langle x, u_{k,l}^{(p)} \rangle$ are zero-mean random variables. A complete description of the second-order statistics of $x[n]$ would generally involve *all* coefficient correlations $\mathrm{E}\big\{\alpha_{k,l}^{(p)} \alpha_{k',l'}^{(p')*}\big\}$. In contrast, robuston processing uses only the *robuston correlations* $r_{k,k';l}$ defined as the average of $\mathrm{E}\big\{\alpha_{k,l}^{(p)} \alpha_{k',l}^{(p)*}\big\}$ (same l, same p) over $p = 0, \cdots, P-1$,

$$r_{k,k';l} := \frac{1}{P} \sum_{p=0}^{P-1} \mathrm{E}\big\{\alpha_{k,l}^{(p)} \alpha_{k',l}^{(p)*}\big\}. \quad (6)$$

Thus, $r_{k,k';l}$ can be interpreted as an integral measure of the correlation of the two robustons $x_{k,l}[n]$ and $x_{k',l}[n]$ located in the kth and k'th time block and the lth frequency band. In fact, one can show

$$r_{k,k';l} = \frac{1}{P} \mathrm{E}\big\{\langle x_{k,l}, \mathbf{S}_{k-k'} x_{k',l} \rangle\big\}.$$

For $k' = k$, we obtain $r_{k,k;l} = \bar{E}_{k,l}/P$ with the *mean robuston energy* $\bar{E}_{k,l} := \mathrm{E}\big\{\|x_{k,l}\|^2\big\} = \sum_{p=0}^{P-1} \mathrm{E}\big\{|\alpha_{k,l}^{(p)}|^2\big\}$.

Estimation. An unbiased estimator of the robuston correlation $r_{k,k';l}$ using a *single* realization of $x[n]$ is given by (cf. (6))

$$\hat{r}_{k,k';l} = \frac{1}{P} \sum_{p=0}^{P-1} \alpha_{k,l}^{(p)} \alpha_{k',l}^{(p)*} = \frac{1}{P} \langle x_{k,l}, \mathbf{S}_{k-k'} x_{k',l} \rangle. \quad (7)$$

This estimate is more stable than the estimate $\alpha_{k,l}^{(p)} \alpha_{k',l}^{(p)*}$ of an individual coefficient correlation $\mathrm{E}\big\{\alpha_{k,l}^{(p)} \alpha_{k',l}^{(p)*}\big\}$ because it uses averaging over P orthogonal LCB components (*intra-subspace averaging*). In particular, if $\alpha_{k,l}^{(p)}$ and $\alpha_{k',l}^{(p')}$ are statistically independent for $p' \neq p$ and $\alpha_{k,l}^{(p)} \alpha_{k',l}^{(p)*}$ has the same variance for $p = 0, \cdots, P-1$, the estimation va riance is reduced by a factor of P. Thus, the statistical descriptors used by robuston signal processing can be estimated with improved stability. (Of course, additional averaging can be used in all estimates if several realizations of $x[n]$ are available.)

4. ROBUSTON PROCESSES

Definition and expressions. The robuston correlations $r_{k,k';l}$ provide a second-order description of $x[n]$ that is incomplete in general (though sufficient for robuston signal processing). This description becomes complete if $x[n]$ is a *robuston process* (RP) that is defined by the following properties of the (zero-mean) coefficients $\alpha_{k,l}^{(p)}$:

- $\alpha_{k,l}^{(p)}$ and $\alpha_{k',l'}^{(p')}$ at different frequencies (i.e., $(l', p') \neq (l, p)$) are uncorrelated.
- $\alpha_{k,l}^{(p)}$ and $\alpha_{k',l}^{(p)}$ at the same frequency (same (l, p)) have equal correlation for all p, i.e., $\mathrm{E}\big\{\alpha_{k,l}^{(p)} \alpha_{k',l}^{(p)*}\big\} = r_{k,k';l}$ for $p = 0, \cdots, P-1$.

These properties can be summarized as

$$\mathrm{E}\big\{\alpha_{k,l}^{(p)} \alpha_{k',l'}^{(p')*}\big\} = r_{k,k';l} \, \delta_{l,l'} \, \delta_{p,p'}. \quad (8)$$

(Note that because $\mathrm{E}\{\alpha_{k,l}^{(p)}\alpha_{k',l}^{(p)*}\}$ was assumed equal for all p, the expression $r_{k,k';l} = \mathrm{E}\{\alpha_{k,l}^{(p)}\alpha_{k',l}^{(p)*}\}$ is consistent with our previous definition of $r_{k,k';l}$ in (6).) From (8) and (2), the cross-correlation of two robustons $x_{k,l}[n]$ and $x_{k',l'}[n]$ of an RP is readily obtained as

$$\mathrm{E}\{x_{k,l}[n]\,x_{k',l'}^*[n']\} = r_{k,k';l}\,P_{k,k';l}[n,n']\,\delta_{l,l'}\,,$$

where $P_{k,k';l}[n,n'] = \sum_{p=0}^{P-1} u_{k,l}^{(p)}[n]\,u_{k',l}^{(p)*}[n']$ is the kernel of the operator $\mathbf{S}_{k-k'}\mathbf{P}_{k',l}$. This again shows that $r_{k,k';l}$ describes the correlation of $x_{k,l}[n]$ and $x_{k',l}[n]$; furthermore, robustons in different frequency bands ($l \neq l'$) are uncorrelated. Finally, with (1) the correlation $R[n,n'] := \mathrm{E}\{x[n]x^*[n']\}$ of an RP can be calculated. The associated correlation operator (whose kernel is $R[n,n']$) is obtained as

$$\mathbf{R} = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{L-1}\mathbf{R}_{k,l}\,, \quad \text{with } \mathbf{R}_{k,l} = \sum_{k'=k-M}^{k+M} r_{k,k';l}\,\mathbf{S}_{k-k'}\mathbf{P}_{k',l}\,. \qquad (9)$$

Here, M denotes the robuston correlation length (i.e., $r_{k,k';l} = 0$ for $|k-k'| > M$, where M may be infinite). Comparing with (4), we see that *the correlation operator of an RP has the structure of an RF,* with the filter coefficients $h_{k,k';l}$ given by the robuston correlations $r_{k,k';l}$. This structural equivalence of RFs and RPs has important implications for robuston signal processing, as will be seen presently.

Properties. In the following summary of properties of RPs, we will use the robuston correlation matrices $\underline{\mathrm{R}}_l$ defined by $[\underline{\mathrm{R}}_l]_{k,k'} = r_{k,k';l}$. Note that $\underline{\mathrm{R}}_l^H = \underline{\mathrm{R}}_l$ because $r_{k',k;l}^* = r_{k,k';l}$.

P_1: A *stationary white process* (i.e., $\mathbf{R} = \sigma^2\mathbf{I}$) is an RP with robuston correlations $r_{k,k';l} = \sigma^2\delta_{k,k'}$ or $\underline{\mathrm{R}}_l = \sigma^2\underline{\mathrm{I}}$ for all l.

P_2: A *weighted sum* $x[n] = ax^{(1)}[n] + bx^{(2)}[n]$ (with a,b nonrandom) of two uncorrelated RPs $x^{(1)}[n]$ and $x^{(2)}[n]$ is an RP with $r_{k,k';l} = |a|^2 r_{k,k';l}^{(1)} + |b|^2 r_{k,k';l}^{(2)}$ or $\underline{\mathrm{R}}_l = |a|^2\underline{\mathrm{R}}_l^{(1)} + |b|^2\underline{\mathrm{R}}_l^{(2)}$.

P_3: An *innovations filter* for an RP $x[n]$ (i.e., a system $\mathbf{H}$ satisfying $\mathbf{HH}^+ = \mathbf{R}$) is given by any RF whose coefficient matrices $\underline{\mathrm{H}}_l$ satisfy $\underline{\mathrm{H}}_l\underline{\mathrm{H}}_l^H = \underline{\mathrm{R}}_l$.

P_4: A *noise whitening filter* for an RP $x[n]$ (i.e., a system $\mathbf{H}$ satisfying $\mathbf{HRH}^+ = \mathbf{I}$) is given by any RF whose coefficient matrices $\underline{\mathrm{H}}_l$ satisfy $\underline{\mathrm{H}}_l\underline{\mathrm{R}}_l\underline{\mathrm{H}}_l^H = \underline{\mathrm{I}}$.

5. APPLICATION TO SIGNAL ESTIMATION

As an example illustrating the application of the robuston scheme in statistical signal processing, we now consider nonstationary signal estimation. Let $s[n]$ and $v[n]$ be mutually uncorrelated, nonstationary signal and noise processes with correlation operator $\mathbf{R}^{(s)}$ and $\mathbf{R}^{(v)}$, respectively. We wish to estimate $s[n]$ from the observed (noisy) signal $x[n] = s[n] + v[n]$ by means of a linear, time-varying filter $\mathbf{H}$. The filter minimizing the mean-square error (MSE) $\varepsilon = \mathrm{E}\{\|\hat{s} - s\|^2\}$ with $\hat{s}[n] = (\mathbf{H}x)[n]$ is given by the equation $\mathbf{HR}^{(x)} = \mathbf{R}^{(s)}$ whose solution is the *nonstationary Wiener filter* [5, 6]

$$\mathbf{H}^{\mathrm{W}} = \mathbf{R}^{(s)}\mathbf{R}^{(x)-1}, \quad \text{with } \mathbf{R}^{(x)} = \mathbf{R}^{(s)} + \mathbf{R}^{(v)}. \qquad (10)$$

A robuston-type Wiener filter can be obtained by two alternative approaches that will be seen to yield essentially the same result.

Wiener filter for robuston processes. In the first approach, we model $s[n]$ and $v[n]$ as uncorrelated RPs with robuston correlation matrices $\underline{\mathrm{R}}_l^{(s)}$ and $\underline{\mathrm{R}}_l^{(v)}$, respectively. Using the structural equivalence of RPs and RFs (see Section 4) and the RF properties P_3–P_5 from Section 2, it then follows that the Wiener filter in (10) is an RF with coefficient matrices

$$\underline{\mathrm{H}}_l^{\mathrm{W}} = \underline{\mathrm{R}}_l^{(s)}\underline{\mathrm{R}}_l^{(x)-1}, \quad \text{with } \underline{\mathrm{R}}_l^{(x)} = \underline{\mathrm{R}}_l^{(s)} + \underline{\mathrm{R}}_l^{(v)}. \qquad (11)$$

Indeed, because $\mathbf{R}^{(s)}$ and $\mathbf{R}^{(v)}$ are RFs, also $\mathbf{R}^{(x)} = \mathbf{R}^{(s)} + \mathbf{R}^{(v)}$ and, in turn, $\mathbf{R}^{(x)-1}$ and $\mathbf{H}^{\mathrm{W}} = \mathbf{R}^{(s)}\mathbf{R}^{(x)-1}$ are RFs. Note that the RF structure of $\mathbf{H}^{\mathrm{W}}$ is a direct consequence of the RP structure of $s[n]$ and $v[n]$; no *a priori* assumption that $\mathbf{H}$ is an RF was used. The coefficient equations corresponding to (11) read

$$\sum_{\kappa=-\infty}^{\infty} h_{k,\kappa;l}\,r_{\kappa,k';l}^{(x)} = r_{k,k';l}^{(s)}\,, \quad k,k' \in \mathbb{Z}\,, \qquad (12)$$

with $r_{k,k';l}^{(x)} = r_{k,k';l}^{(s)} + r_{k,k';l}^{(v)}$.

Thus, we have obtained a nonstationary Wiener filter that is an RF and whose design only requires knowledge of the robuston correlations $r_{k,k';l}^{(s)}$ and $r_{k,k';l}^{(v)}$. We finally note that the minimum MSE achieved with $\mathbf{H}^{\mathrm{W}}$ can be calculated as $\varepsilon_{\min} = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{L-1}\varepsilon_{k,l}^{\min}$, with $\varepsilon_{k,l}^{\min}$ the kth diagonal element of the matrix $P\underline{\mathrm{R}}_l^{(s)}\underline{\mathrm{R}}_l^{(x)-1}\underline{\mathrm{R}}_l^{(v)}$.

Optimal robuston filter for general processes. In the second approach, we do not assume an RP structure for $s[n]$ and $v[n]$ but we constrain $\mathbf{H}$ to be an RF of the form (4) with given length parameters M_1, M_2. The RF coefficients $h_{k,k';l}$ minimizing the MSE ε can be derived as follows. The MSE allows the decomposition

$$\varepsilon = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{L-1}\varepsilon_{k,l}\,, \quad \text{with } \varepsilon_{k,l} = \mathrm{E}\{\|\hat{s}_{k,l} - s_{k,l}\|^2\}\,,$$

where $\hat{s}_{k,l}[n]$ and $s_{k,l}[n]$ are the robustons of $\hat{s}[n]$ and $s[n]$, respectively. From

$$\hat{s}_{k,l}[n] = \sum_{k'=k-M_1}^{k+M_2} h_{k,k';l}\,(\mathbf{S}_{k-k'}\mathbf{P}_{k',l}\,x)[n] \qquad (13)$$

we see that $\varepsilon_{k,l}$ depends only on the coefficients $h_{k,k';l}$ and not on other coefficients $h_{\tilde{k},k';\tilde{l}}$. Therefore, each $\varepsilon_{k,l}$ can be minimized *separately* with respect to the associated $h_{k,k';l}$. Due to (13) and the orthogonality principle [5, 6], each robuston error component $\hat{s}_{k,l}[n] - s_{k,l}[n]$ must satisfy $\mathrm{E}\{\langle\hat{s}_{k,l} - s_{k,l}\,,\,\mathbf{S}_{k-k'}\mathbf{P}_{k',l}\,x\rangle\} = 0$ for $k' \in [k-M_1, k+M_2]$. With (13), this yields the set of equations

$$\sum_{\kappa=k-M_1}^{k+M_2} h_{k,\kappa;l}\,r_{\kappa,k';l}^{(x)} = r_{k,k';l}^{(s)}\,, \quad k \in \mathbb{Z},\ k' \in [k-M_1, k+M_2]\,, \qquad (14)$$

with $r_{k,k';l}^{(x)} = r_{k,k';l}^{(s)} + r_{k,k';l}^{(v)}$. Remarkably, calculation of the optimal RF requires only the robuston correlations $r_{k,k';l}^{(s)}$ and $r_{k,k';l}^{(v)}$ although $s[n]$ and $v[n]$ were *not* assumed to be RPs. We can write (14) as the system of equations of size $(M_1+M_2+1)\times(M_1+M_2+1)$

$$\underline{\mathrm{R}}_{k,l}^{(x)}\,\underline{\mathrm{h}}_{k,l} = \underline{\mathrm{r}}_{k,l}^{(s)}\,,$$

with $[\underline{\mathrm{R}}_{k,l}^{(x)}]_{m,m'} = r_{k+m',k+m;l}^{(x)}$, $[\underline{\mathrm{h}}_{k,l}]_m = h_{k,k+m;l}$, and $[\underline{\mathrm{r}}_{k,l}^{(s)}]_m = r_{k,k+m;l}^{(s)}$ ($m,m' \in [-M_1, M_2]$). The vector $\underline{\mathrm{h}}_{k,l}$ contains the $M_1 + M_2 + 1$ RF coefficients for the robuston index (k,l); it is given by

$$\underline{\mathrm{h}}_{k,l}^{\mathrm{opt}} = \underline{\mathrm{R}}_{k,l}^{(x)-1}\underline{\mathrm{r}}_{k,l}^{(s)}\,. \qquad (15)$$

The resulting minimum MSE is given by $\varepsilon_{\min} = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{L-1}\varepsilon_{k,l}^{\min}$, where $\varepsilon_{k,l}^{\min} = \bar{E}_{k,l}^{(s)} - P\,\underline{\mathrm{r}}_{k,l}^{(s)H}\,\underline{\mathrm{R}}_{k,l}^{(x)-1}\underline{\mathrm{r}}_{k,l}^{(s)}$ with $\bar{E}_{k,l}^{(s)} := \mathrm{E}\{\|s_{k,l}\|^2\}$.

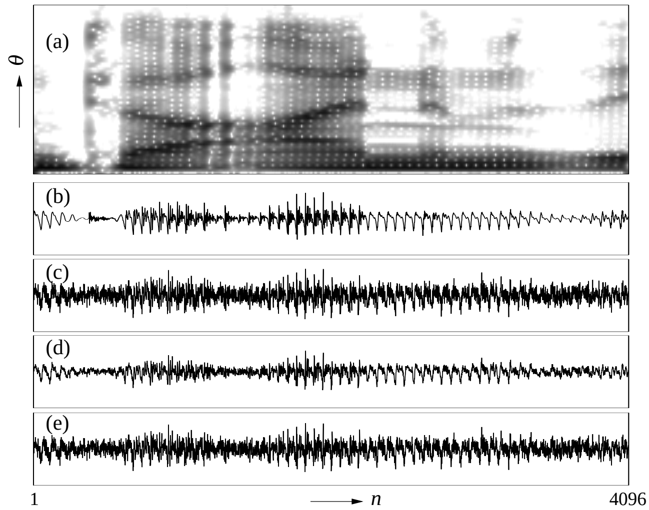

Figure 2: *Estimation of a speech signal: (a) Spectrogram of signal $s[n]$, (b) signal $s[n]$, (c) noisy signal $x[n] = s[n] + \nu[n]$ for an SNR of 0 dB, (d) estimate $\hat{s}[n]$ obtained with RF $\mathbf{H}^{\text{opt}}$ ($M_1 = M_2 = 1$, $N = 256$, $P = 32$), (e) estimate $\hat{s}[n]$ obtained with SWF $\mathbf{H}^{\text{SWF}}$.*

It is interesting to note that the equations (14) are equivalent to (12) except for the finite filter length and the finite k' range. For $M_1 = M_2 = \infty$, our two approaches become altogether equivalent. For $M_1 = M_2 = 0$, on the other hand, the optimal RF becomes

$$\mathbf{H}^{\text{opt}} = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{L-1} h_{k,l}^{\text{opt}} \mathbf{P}_{k,l} \qquad \text{for } M_1 = M_2 = 0\,,$$

where $h_{k,l}^{\text{opt}} = r_{k,k;l}^{(s)} / r_{k,k;l}^{(x)} = \bar{E}_{k,l}^{(s)} / \bar{E}_{k,l}^{(x)} = \bar{E}_{k,l}^{(s)} / \big(\bar{E}_{k,l}^{(s)} + \bar{E}_{k,l}^{(\nu)}\big)$. This filter was previously shown to be minimax robust with respect to specific uncertainty classes for the correlations of $s[n]$ and $\nu[n]$ [1,2].

Simulation results. To assess the performance of robuston signal processing, we apply the optimal RF $\mathbf{H}^{\text{opt}}$ in (15) to the estimation of a speech signal. We used a recorded speech signal of length 4096 samples as a realization $s[n]$ of a nonstationary signal process with unknown statistics (thus, $s[n]$ is *not* the realization of an RP). The noise $\nu[n]$ was a realization of a stationary and white process with known variance σ_ν^2. The signal $s[n]$ and its noisy version $x[n] = s[n] + \nu[n]$ (for an SNR of 0 dB) are shown in Fig. 2(a)–(c).

For designing $\mathbf{H}^{\text{opt}}$, an estimate $\hat{r}_{k,k';l}^{(x)}$ of $r_{k,k';l}^{(x)}$ was calculated from $x[n]$ according to (7), and an estimate of $r_{k,k';l}^{(s)}$ was then obtained as $\hat{r}_{k,k';l}^{(s)} = \big[\hat{r}_{k,k';l}^{(x)} - \sigma_\nu^2\, \delta_{k,k'}\big]_+$ (corresponding to the positive semidefinite part of the matrix $\underline{\hat{\mathbf{R}}}_l^{(s)} = \underline{\hat{\mathbf{R}}}_l^{(x)} - \sigma_\nu^2 \underline{\mathbf{I}}$). The RF used filter lengths $M_1 = M_2 = 1$ (i.e., total filter length $M_1 + M_2 + 1 = 3$), block length $N = 256$, and robuston dimensions $P \in \{1, 2, 4, 8, \cdots, 256\}$.

For comparison, we also considered an RF with $M_1 = M_2 = 0$ (i.e., total filter length 1) and $P = 1$ (i.e., no averaging over subbands). Here, each single subband signal sample (i.e., LCB expansion coefficient $\alpha_{k,l}^{(0)} = \langle x, u_{k,l}^{(0)} \rangle$) is separately weighted by $h_{k,l} = \hat{P}_{k,l}^{(s)} / \hat{P}_{k,l}^{(x)}$, with estimated subband sample powers $\hat{P}_{k,l}^{(x)} = |\alpha_{k,l}^{(0)}|^2$ and $\hat{P}_{k,l}^{(s)} = \big[|\alpha_{k,l}^{(0)}|^2 - \sigma_\nu^2\big]_+$ (which is $|\alpha_{k,l}^{(0)}|^2 - \sigma_\nu^2$ if this number is positive and 0 otherwise). This filter, hereafter denoted by $\mathbf{H}^{\text{SWF}}$, can be viewed as a simple on-line "subband Wiener filter" (SWF) that does not exploit the correlations of temporally adjacent $\alpha_{k,l}^{(0)}$ and does not employ intra-subspace averaging. Therefore, our comparison $\mathbf{H}^{\text{opt}}$ vs. $\mathbf{H}^{\text{SWF}}$ shows the effect of temporal filtering ($M_1 + M_2 + 1 > 1$) and subspace averaging ($P > 1$) on the estimation performance. Note that we do not consider the full-blown Wiener filter $\mathbf{H}^{\text{W}}$ in (10) because the computational complexity of its design and implementation would be excessive for the given signal length of 4096 samples. Figs. 2(d) and (e) show the signal estimates $\hat{s}[n]$ obtained with $\mathbf{H}^{\text{opt}}$ (with $M_1 = M_2 = 1$, $N = 256$, $P = 32$, and $L = 8$) and $\mathbf{H}^{\text{SWF}}$. Clearly, the result of $\mathbf{H}^{\text{opt}}$ is much better than that of $\mathbf{H}^{\text{SWF}}$.

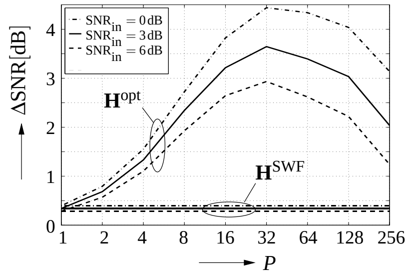

Figure 3: *SNR improvement using $\mathbf{H}^{\text{opt}}$ ($M_1 = M_2 = 1$, $N = 256$) vs. subspace dimension P, for three different input SNRs. For comparison, also the SNR improvement using $\mathbf{H}^{\text{SWF}}$ is shown.*

For a more complete performance comparison and analysis, we repeated the experiment described above 40 times, using the same speech signal $s[n]$ but different noise signals $\nu[n]$. Fig. 3 shows the SNR improvement (averaged over the 40 realizations) obtained with $\mathbf{H}^{\text{opt}}$ vs. the subspace dimension P, for three different input SNRs. For all three input SNRs, the maximum SNR improvement is obtained for $P = 32$ (e.g., 4.4 dB at input SNR 0 dB). For comparison, also the SNR improvement obtained with $\mathbf{H}^{\text{SWF}}$ is plotted (recall that $\mathbf{H}^{\text{SWF}}$ uses $P = 1$ and $M_1 = M_2 = 0$). It is seen that $\mathbf{H}^{\text{opt}}$ outperforms $\mathbf{H}^{\text{SWF}}$ by up to about 4 dB. These results demonstrate the potential performance advantages of robuston signal processing.

6. CONCLUSIONS

We have introduced a new paradigm for nonstationary signal processing in which subspace signal components (called robustons) are used as elementary atomic entities. The resulting reduced-detail signal modeling and processing methods employ intra-subspace averaging to estimate the relevant statistics with improved stability. Robuston signal processing allows efficient on-line implementations with inherent localization in time and frequency. The performance advantages of robuston signal processing were demonstrated for a nonstationary signal estimation application.

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