Tracking Analysis of Normalized Adaptive Algorithms

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Abstract

In this work, tracking analysis of normalized adaptive algorithms is carried out in the presence of two sources of nonstationarites; carrier frequency offset between transmitter and receiver, and random variations in the environment. A unified approach is carried out using a mixed-norm-type error nonlinearity. Close agreement between analytical analysis and simulation results is obtained for the case of the NLMS algorithm, which is the only algorithm considered here due to space limitations. The results show that unlike the stationary case, the steady-state excess-mean-square error is not a monotonically increasing function of the step-size, while the ability of the adaptive algorithm to track the variations in the environment degrades by increasing the frequency offset.

1 Introduction

Cyclic and random system nonstationarities are common impairment in communication systems and especially in applications that involve channel estimation, channel equalization, and inter-symbol-interference cancellation. Random nonstationarity is present due to variations in channel characteristics which is true in most of the cases particularly in the case of mobile communication environment [1]. Cyclic system nonstationarities arise in communication systems due to mismatches between the transmitter and receiver carrier generator. The ability of adaptive filtering algorithms to track such systems variations are not fully understood. In this regard, a recent contribution [2] presented a first order analysis of the performance of the Least Mean Squares (LMS) algorithm [3] in the presence of the carrier frequency offset. In [4], a general framework for the tracking analysis of adaptive algorithms was developed that can handle both cyclic as well as random system nonstationarities simultaneously. The framework, based on energy conversation relation [5], holds for all adaptive algorithms whose recursion are of the form:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu \mathbf{x}_n f(e_n). \qquad (1)$$

where $f(e_n)$ denotes a general scalar function of the output estimation error e_n, μ is the step-size used in adaptation of filter coefficients, and $\mathbf{w}_n$ is the vector representing the coefficients of the adaptive filter.

In so far work, the analysis of adaptive algorithms has been restricted to only non-normalized adaptive algorithms. This work presents an extension to normalized versions, i.e., the normalized LMS algorithm [6] and the normalized LMF algorithm [7], where a unified analysis is carried out for the case where the nonlinearity $f(e_n)$ is of the form:

$$f(e_n) = h(e_n) g(\mathbf{x}_n) \qquad (2)$$

where $h(e_n)$ is a purely error nonlinearity and $g(\mathbf{x}_n)$ is a purely input nonlinearity. More specifically, the error nonlinearity considered here is of a mixed-norm type format [8]. Under this case, the input nonlinearity is defined as :

$$g(\mathbf{x}_n) = \frac{\mathbf{I}}{\| \mathbf{x}_n \|^2} \qquad (3)$$

and the error nonlinearity is defined as:

$$h(e_n) = \alpha_n e_n + 2(1-\alpha_n) e_n^3 \qquad (4)$$

where $\mathbf{I}$ is the identity matrix, $\| \mathbf{x}_n \|^2$ is the Eucledian norm of the input sequence $\{x_n\}$, and α_n is a time-varying mixing parameter in the range [0, 1] so that the unimodal character of the cost function minimized for this purpose is preserved, and it is updated as follows [9]:

$$\begin{aligned} \alpha_{n+1} &= \delta \alpha_n + \gamma |p|_n^2, && (5) \\ p_n &= \beta p_{n-1} + (1-\beta) e_n e_{n-1}^*, && (6) \end{aligned}$$

with δ, β, and γ as constants, and * denotes complex conjugate operation. The parameters δ and β, confined to the interval [0,1], are exponential weighting parameters that govern the averaging time constant, i.e., the quality of estimation, and $\gamma > 0$.

2 System Model and Performance Measure

In this section a general system model is presented which includes both types of nonstationarities, that is random and cyclic nonstationarities.
To start, consider the noisy measurement d_n that arises from a model of the form:

$$d_n = \mathbf{x}_n^T \mathbf{w}_n^o e^{j\Omega n} + \xi_n, \tag{7}$$

where ξ_n is the measurement noise and $\mathbf{w}_n^o$ is the unknown system that is to be tracked. The multiplicative term $e^{j\Omega n}$ accounts for a possible frequency offset between the transmitter and receiver carriers in a digital communication scenario. Furthermore it is assumed that the unknown system vector $\mathbf{w}_n^o$ is randomly changing according to:

$$\mathbf{w}_n^o = \mathbf{w}^o + \mathbf{q}_n. \tag{8}$$

where $\mathbf{w}^o$ is a fixed vector, and $\mathbf{q}_n$ is assumed to be a zero-mean stationary random vector process with a positive definite covariance matrix $\mathbf{Q}_n = E[\mathbf{q}_n \mathbf{q}_n^T]$. Moreover, it is also assumed that the sequence $\{q_n\}$ is statistically independent of the sequences $\{\xi_n\}$ and $\{x_n\}$. Thus, from the generalized system model given by Equations (7) and (8), it can be seen that the effects of both cyclic and random system nonstationarities are included in this system model.
In the steady-state analysis of adaptive algorithms, an important measure of performance is their steady-state mean square error (MSE), which is defined as:

$$\begin{aligned} \text{MSE} &= \lim_{n\to\infty} E[e_n^2] && (9) \\ &= \lim_{n\to\infty} E\{[\xi_n + \mathbf{x}_n^T \mathbf{v}_n]^2\}. && (10) \end{aligned}$$

where $\mathbf{v}_n$ is the weight error vector defined as:

$$\mathbf{v}_n = \mathbf{w}_n^o e^{j\Omega n} - \mathbf{w}_n. \tag{11}$$

Also of interest, is the steady-state excess-mean-square-error (EMSE), denoted by ζ, and is given by:

$$\zeta = \lim_{n\to\infty} E\{[\mathbf{x}_n^T \mathbf{v}_n]^2\}. \tag{12}$$

3 Fundamental Energy Relation

The fundamental energy conservation relation [4] is presented next. Using Equation (1) and Equation (8), the following recursion is obtained:

$$\mathbf{v}_{n+1} = \mathbf{v}_n - \mu \mathbf{x}_n^* f(e_n) + \mathbf{c}_n e^{j\Omega n}, \tag{13}$$

where $\mathbf{c}_n$ is defined as:

$$\mathbf{c}_n = \mathbf{w}^o (e^{j\Omega} - 1) + \mathbf{q}_{n+1} e^{j\Omega} - \mathbf{q}_n. \tag{14}$$

Now, let's define the following so-called a priori estimation error, $e_{an} = \mathbf{x}_n^T \mathbf{v}_n$ and a posteriori estimation error, $e_{pn} = \mathbf{x}_n^T (\mathbf{v}_{n+1} - \mathbf{c}_n e^{j\Omega n})$. Then, it is very easy to show that the estimation error and the a priori error are related via $e_n = e_{an} + \xi_n$. Also, the a posteriori error is defined in terms of the a priori error as follows:

$$e_{pn} = e_{an} - \frac{\mu}{\hat{\mu}_n} f(e_n). \tag{15}$$

Substituting Equation (15) into Equation(13) results into the following update relation:

$$\mathbf{v}_{n+1} = \mathbf{v}_n - \hat{\mu}_n \mathbf{x}_n^* [e_{an} - e_{pn}] + \mathbf{c}_n e^{j\Omega n}. \tag{16}$$

By evaluating the energies of both sides of the above equation, the following relation is obtained:

$$\| \mathbf{v}_{n+1} - \mathbf{c}_n e^{j\Omega n} \|^2 + \hat{\mu}_n \| e_{an} \|^2 = \| \mathbf{v}_n \|^2 + \hat{\mu}_n \| e_{pn} \|^2 . \tag{17}$$

It can be seen that if $\Omega = 0$ (i.e., no frequency offset between the transmitter and the receiver), the above equation reduces to the basic fundamental energy relation.

4 Tracking Analysis

The energy relation (17) will be used to evaluate the excess-mean-square error at steady state. But before starting the analysis, first the following assumptions are stated:

A1 In steady-state, the weight error vector $\mathbf{v}_n$ takes the generic form $\mathbf{z}_n e^{j\Omega n}$, with the stationary random process $\mathbf{z}_n$ independent of the frequency offset Ω.

A2 The noise ξ_n is a zero-mean iid process, and is independent of the input process. This assumption is justified in several practical examples.

Using Equation (15), assumption **A1**, and taking expectation of both sides of Equation (17) and the fact that at steady state $E[\mathbf{v}_{n+1}] = E[\mathbf{v}_n]$, the following relation can be obtained:

$$\begin{aligned} E[\hat{\mu}_n \| e_{an} \|^2] = {} & 2tr\{\mathbf{Q}_n\} + \| \mathbf{w}^o \|^2 \, |1 - e^{j\Omega}|^2 \\ & -2Re\{E[\mathbf{q}_n^* (\mathbf{z}_n - \mu \mathbf{x}_n^* f(e_n) e^{-j\Omega n})]\} \\ & -2Re\{(1 - e^{j\Omega})^* \mathbf{w}^{o*} \\ & \times E[\mathbf{z}_n - \mu \mathbf{x}_n^* f(e_n) e^{-j\Omega n}]\} \\ & + E\Big[\hat{\mu}_n |e_{an} - \frac{\mu}{\hat{\mu}_n} f(e_n)|^2\Big], \end{aligned} \tag{18}$$

which can be used to solve for the steady-state excess-mean-square error (EMSE).

To find the value of $\mathbf{z} = E[\mathbf{z}_n]$, Equation (13) is used where it is multiplied by the term $e^{-j\Omega n}$ and then expectation is taken on both sides to get:

$$(1-e^{j\Omega})\mathbf{z} = \mu E\Big(\mathbf{x}_n^* f(e_n) e^{-j\Omega n}\Big) + \mathbf{w}^o(1-e^{j\Omega}), \tag{19}$$

which yields the value of $\mathbf{z}$ at steady-state:

$$\begin{aligned}\mathbf{z} &= \frac{\mu}{(1-e^{j\Omega})}\Bigg[E[\alpha_n] + 6(1-E[\alpha_n])\sigma_w^2\Bigg]\frac{\mathbf{R}}{tr\{\mathbf{R}\}}\mathbf{z} \\ &\quad +\mathbf{w}^o \\ &= \Bigg[\mathbf{I} - \frac{\mu\gamma_o}{(1-e^{j\Omega})}\mathbf{R}\Bigg]^{-1}\mathbf{w}^o,\end{aligned} \tag{20}$$

where γ_o is defined as:

$$\gamma_o = \Bigg[E[\alpha_n] + 6E[\bar{\alpha_n}]\sigma_w^2\Bigg]\frac{1}{tr\{\mathbf{R}\}}, \tag{21}$$

and $\bar{\alpha_n} = (1-\alpha_n)$.
Ultimately, the steady-state excess-mean-square error of the proposed algorithm, ζ^{prop}, is obtained from Equation (18):

$$\zeta^{prop} = \frac{2p}{2p-\mu a}\Bigg[tr\{\mathbf{Q}_n\mathbf{R}\} + \frac{\beta_{op}}{2\mu\gamma_o} + \frac{b\mu}{2p}\Bigg], \tag{22}$$

where

$$p = E[\alpha_n] + 6(1-E[\alpha_n])\sigma_w^2, \tag{23}$$

$$a = E[\alpha_n^2] + 36E[\bar{\alpha_n}^2]\phi_w^4 + 12E[\alpha_n]E[\bar{\alpha_n}]\sigma_w^2, \tag{24}$$

$$b = E[\alpha_n^2]\sigma_w^2 + 4E[\bar{\alpha_n}^2]\phi_w^6 + 4E[\alpha_n]E[\bar{\alpha_n}]\phi_w^4, \tag{25}$$

and

$$\beta_{op} = \left|1-e^{j\Omega}\right|^2 Re\Bigg\{tr\big(\mathbf{W}^o(\mathbf{I}-2\mathbf{X}_p)\big)\Bigg\}. \tag{26}$$

Accordingly, Equation (22) can be used to get an expression for the steady-state excess-mean-square-error of the NLMS algorithm:

$$\zeta^{NLMS} = \frac{1}{(2\mu-\mu^2)}\Bigg\{2\mu tr\{\mathbf{Q}_n\mathbf{R}\} + \mu^2\sigma_w^2 + \beta_o tr\{\mathbf{R}\}\Bigg\} \tag{27}$$

where

$$\beta_o = \left|1-e^{j\Omega}\right|^2 Re\Big\{tr\big(\mathbf{W}^o(\mathbf{I}-2\mathbf{X})\big)\Big\}, \tag{28}$$

and

$$\mathbf{X} = \Bigg[\mathbf{I} - \mu\frac{\mathbf{R}}{tr\{\mathbf{R}\}}\Bigg]\Bigg[\mathbf{I} - \mu\frac{\mathbf{R}}{tr\{\mathbf{R}\}} - e^{j\Omega}\mathbf{I}\Bigg]^{-1}. \tag{29}$$

For a white Gaussian input signal, the autocorrelation of the input signal $\mathbf{R} = \sigma_x^2\mathbf{I}$, and therefore:

$$tr\{\mathbf{R}\} = N\sigma_x^2, \tag{30}$$

where N is the filter length. Accordingly, Equations (22) and (27), respectively, look like the following:

$$\begin{aligned}\zeta^{prop} &= \frac{2p}{2p-\mu a}\Bigg[\sigma_x^2 tr\{\mathbf{Q}_n\} + \frac{b\mu}{2p} \\ &\quad + \frac{N^2\sigma_x^2(2N-\mu p)\Omega^2}{2\mu^2 p^2}\parallel \mathbf{w}^o \parallel^2\Bigg],\end{aligned} \tag{31}$$

and

$$\begin{aligned}\zeta^{NLMS} &= \frac{2}{2-\mu}\Bigg[\sigma_x^2 tr\{\mathbf{Q}_n\} + \frac{\mu\sigma_w^2}{2} \\ &\quad + \frac{N^2\sigma_x^2(2N-\mu)\Omega^2}{2\mu^2}\parallel \mathbf{w}^o \parallel^2\Bigg].\end{aligned} \tag{32}$$

5 Simulation Results

The simulations are carried out for a system identification problem, where the unknown system, having a FIR model, is given by $[1.0119-j0.7589,\ -0.3796+j0.5059]^T$, while the system characteristics are time-varying. As mentioned earlier, only results for the NLMS algorithm are presented to validate the theoretical findings, that is Equation (32), for different values of Ω and different values of μ.

The input signal x_n to both the unknown system and the adaptive filter is obtained by passing a zero-mean white Gaussian sequence through a channel that is used to vary the eigenvalue spread of the autocorrelation matrix of the input signal. The example considered for the sequence $\{x_n\}$ has an eigenvalue spread of 68.9. The signal to noise ratio is set to be equal to 30 dB and $tr\{\mathbf{Q}_n\} = 10^{-7}$.

Figure 1 depicts the comparison of the theory to the simulation results for three different values of Ω, i.e., $\Omega = 0.01,\ 0.02$, and 0.03. As can be seen from this figure, close agreement between theory and simulation results are obtained. Furthermore, it is observed from this figure that degradation in performance is obtained by increasing the frequency offset Ω and unlike the stationary case, the steady-state EMSE is not a monotonically

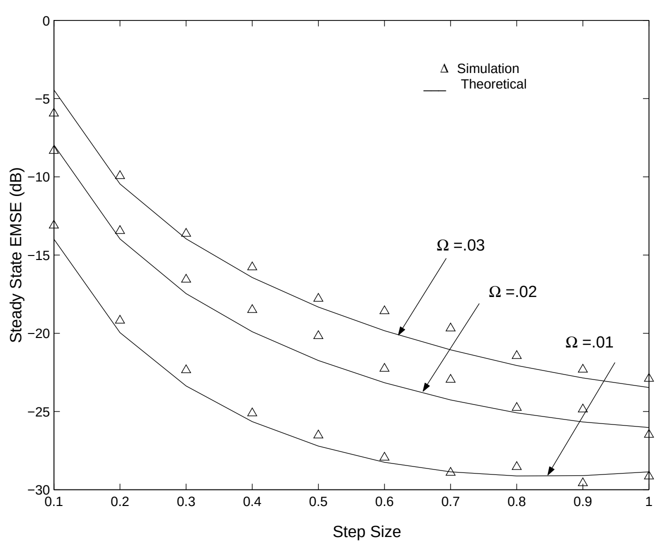

Figure 1: Comparison between Analytical and Experimental ζ_{NLMS} at Ω =0.01, Ω=0.02, and Ω=0.03

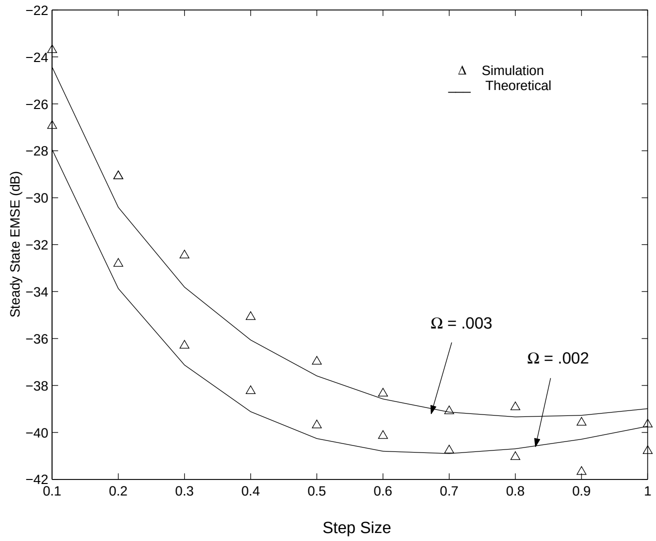

Figure 2: Comparison between Analytical and Experimental ζ_{NLMS} at Ω =0.002 and Ω=0.003

increasing function of the step-size μ, that is the steady-state EMSE is smaller at larger values of the step-size μ. Similar behaviour is observed on Figure 2 for the case of $\Omega = 0.002$ and $\Omega = 0.003$.

Finally, the consistency in the performance of the steady-state excess-mean-square error of the NLMS algorithm is observed on other experiments.

6 Conclusion

The analytical results of the steady-state EMSE are derived for normalized adaptive algorithms in the presence of both random and cyclic nonstationarities. The results, for the case of the NLMS algorithm, show that unlike the stationary case, the steady-state EMSE is not a monotonically increasing function of the step-size μ, while the ability of the adaptive algorithm to track the variations in the environment degrades by increasing the frequency offset Ω.

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