LOW-ORDER IIR FILTER BANK DESIGN

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ABSTRACT

The advantage of IIR filters over FIR ones is that the formers require a much lower order to obtain the desired response specifications. However, the existing deterministic techniques for IIR filter bank design based on heuristic usually lead to too high order IIR filters and thus cannot be practically used. In this paper, we propose new method to solve the low-order IIR filter bank design, which is based on LMI optimization. Our focus is the QMF bank design, although other IIR filter related problems can be treated and solved in similar way.

1. INTRODUCTION

The trade-off between IIR filter and FIR one is that the former requires much lower orders for a given desired specification. The existing deterministic methodologies often lead to too high order IIR filters so the mentioned trade-off between IIR and FIR filters are hardly enhanced. It is especially true in the IIR filter bank design problems[7, 5], which is formulated as follows:
Given two minimum-phase analysis (IIR) filters $H_0(z)$ and $H_1(z)$, design two minimum-phase (synthesis) filters $G_0(z)$ and $G_1(z)$ such that

$$H_0(z)G_0(z) + H_1(z)G_1(z) = z^{-n_0} \textit{ for some } n_0 > 0, \quad (1)$$

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0. \quad (2)$$

A natural optimization formulation for handling constraints (1), (2) is thus

$$\min_{G_0(z),G_1(z)} \quad ||z^{-n_0} - [H_0(z)G_0(z) + H_1(z)G_1(z)]|| : \quad s.t\ (2), \quad (3)$$

where the analysis filters are predesigned. When the norm $||.||$ in (3) is understood as $\mathcal{H}_\infty$-norm defined by

$$\sup_{\omega\in[0,2\pi]} |e^{-\jmath n_0\omega} - [H_0(e^{\jmath\omega})G_0(e^{\jmath\omega}) + H_1(e^{\jmath\omega})G_1(e^{\jmath\omega})]|, \quad (4)$$

we call $G_0(z), G_1(z)$ by $\mathcal{H}_\infty$ filters. Analogously, they will be called by either $\mathcal{H}_2$-filters or mixed-norm filters when the norm $||.||$ in (3) is understood as $\mathcal{H}_2$-norm defined by

$$\frac{1}{\sqrt{2\pi}}[\int_0^{2\pi} |e^{-\jmath n_0\omega} - [H_0(e^{\jmath\omega})G_0(e^{\jmath\omega}) + H_1(e^{\jmath\omega})G_1(e^{\jmath\omega})]|^2 d\omega]^{1/2}, \quad (5)$$

or even as a mixed $\mathcal{H}_\infty/\mathcal{H}_2$ norm, respectively.
Substituting this value into (3), one gets the equivalent optimization problem

$$\min_{G(z)} ||z^{-n_0} - H(z)G(z)||, \quad (6)$$

$$\text{where } H(z) = H_0(z)H_1(-z) - H_1(z)H_0(-z). \quad (7)$$

In the case of $\mathcal{H}_\infty$-filters, problem (6) is a model-matching or Nehari problem [3]. As a result, the optimal solution $G_{opt}(z)$ can be found by classical results of $\mathcal{H}_\infty$ control based on solutions of Riccati equations. The disadvantage of this method is that the order of this optimal solution $G_{opt}(z)$ is too high for practical implementation . On the other hand, it is also obvious that (6) is a particular case of general output feedback control problems. However, the solution is still very conservative because one still has to employ the balanced truncation[8] to reduce the high order of the design filter $G(z)$ and one has to use a single Lyapunov function to handle both $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms to obtain for an attractive LMI formulation[4]. In this paper we show that the following issues are overcome. The first is the high order of designed filters. It will be shown that LMI optimization based formulations for fixed order filters (when $G(z)$ in (6) is restricted on the class of fixed $k-$th order IIR filers) are much more attractive than that of output feedback control, which allow us to solve the problem very effectively. The second is mixed-norm optimization. Generally, minimized energy($\mathcal{H}_2$-filters) and peak($\mathcal{H}_\infty$-filters) are conflict requirements. Thus, a good trade-off is the mixed-norm filter problem which can be described by one of the following ones and/or their combinations

$$\min_{G(z)} \quad [\mu||z^{-n_0} - H(z)G(z)||_\infty + (1-\mu)||z^{-n_0} - H(z)G(z)||_2], \quad (8)$$

$$\min_{G(z)} \quad ||z^{-n_0} - H(z)G(z)||_2 : \quad ||z^{-n_0} - H(z)G(z)||_\infty < \gamma, \quad (9)$$

when $0 < \mu < 1$ is used to express the trade-off between $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norm criteria. Then, each norm criterion in (8) and (9) is derived on its own checking Lyapunov function.
The structure of the paper is as follows. Section 2 summarizes some basic tools used in our approach. Section 3 addresses the optimization problem (6) with $\mathcal{H}_\infty$-norm and $\mathcal{H}_2$-norm separately treated, while the optimization multi-criterion problems (8), (9) are discussed in Section 4. The theoretical development of Sections 3 and 4 are confirmed by extensive simulation in Section 5. Finally, some conclusions are drawn at Section 6.

2. EQUIVALENT STATE SPACE-BASED OPTIMIZATION FORMULATIONS

Our goal in this paper is to provide LMI based attractive formulations for optimization problems (6), (8), (9). This section summarizes several useful formulations and basic tools needed in our derivation.
For an $k-$order IIR filter

$$G(z) = \left(\sum_{i=0}^{k} a_i z^{-i}\right)\left(1+\sum_{i=1}^{k} b_i z^{-i}\right)^{-1} \tag{10}$$

there are matrices A_F, B_F, L_F, M_F of dimension $k\times k, k\times 1,\ 1\times k, 1\times 1$, used in the state-space representation of $G(z)$ such that $G(z) = L_F(zI - A_F)^{-1}B_F + M_F$, and vice-versa. For short, it is custom to write

$$G(z) = \left[\begin{array}{c|c} A_F & B_F \\ \hline L_F & M_F \end{array}\right] \tag{11}$$

Consequently, by introducing the state variable $x_i^F \in R^k$, the relation $Z^F(z) = G(z)Y(z)$ in $z-$domain can be expressed in state-space setting by the following linear time invariant (LTI) system

$$\begin{array}{rcl} x_{i+1}^F &=& A_F x_i^F + B_F y_i, \qquad A_F \in \boldsymbol{R}^{k\times k}, B_F \in \boldsymbol{R}^{k\times 1}, \\ z_i^F &=& L_F x_i^F + M_F y_i,\ L_F \in \boldsymbol{R}^{1\times k},\ M_F \in \boldsymbol{R}, \end{array} \tag{12}$$

where z_i^F and y_i are time representation of $Z^F(z)$ and $Y(z)$, respectivelly. Now, take any minimal state space realization of z^{-n_0} and $H(z)$ as

$$\begin{bmatrix} H(z) \\ z^{-n_0} \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ L & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_H & 0 & B_H \\ 0 & A_0 & B_0 \\ \hline C_H & 0 & D_H \\ 0 & C_0 & 0 \end{array}\right], \tag{13}$$

which is also written as

$$\begin{array}{rcll} x_{i+1} &=& Ax_i + Bw_i, & A \in \boldsymbol{R}^{n\times n},\ B \in \boldsymbol{R}^{n\times 1}, \\ y_i &=& Cx_i + Dw_i, & C \in \boldsymbol{R}^{1\times n},\ D \in \boldsymbol{R} \\ z_i &=& Lx_i & L \in \boldsymbol{R}^{1\times n} \end{array} \tag{14}$$

where $x_i \in R^n$ is the newly introduced state variable, $y_i \in R$ is the time-domain transform of $Y(z) = H(z)W(z)$ and so is treated as the measured output while $z_i \in R$ is the time-domain transform of $Z(z) = z^{-n_0}W(z)$ and thus is treated as the output to be estimated/tracked. Using (12) and (14), the state-space representation of $z^{-n_0} - H(z)G(z)$ in (6) is the input-output LTI system

$$\begin{array}{rcl} x_{i+1} &=& Ax_i + Bw_i,\ y_i = Cx_i + Dw_i,\ z_i = Lx_i; \\ x_{i+1}^F &=& A_F x_i^F + B_F y_i,\ z_i^F = L_F x_i^F + M_F y_i, \\ z_i^{cl} &=& z_i - z_i^F \end{array} \tag{15}$$

In summary, solving any problem in (6), (8), (9), (12) requires one to find A_F, B_F, L_F, M_F to minimize either $\mathcal{H}_2$ or $\mathcal{H}_\infty$ or mixed norms of LTI system (15) or system (12). Without loss of generality, we set $M_F = 0$. The $\mathcal{H}_2$-norm or $\mathcal{H}_\infty$-norm of LTI systems will be discussed in the next section.

3. FIXED ORDER OPTIMAL IIR SYNTHESIS FILTERS

3.1. Optimal $\mathcal{H}_2$ synthesis filter

By using the state-space representation (15), formula (5) for $\mathcal{H}_2$-norm is also interpreted as the variance $\mathcal{E}(z_{cl}^2)$ of z^{cl} in (15) under the assumption that w_i are white noise with unit variance. Therefore, with $\mathcal{H}_2$-norm in (6) used as our minimization criterion and $G(z)$ is restricted in the class of fixed k-order filters, the problem is then to find the matrices A_F, B_F, L_F of dimension $k\times k,\ k\times 1,\ 1\times k$ such that the variance $\mathcal{E}(z_{cl}^2)$ of z^{cl} in (15) is minimized. The main result of this section is sketched in the following theorem:

Theorem 1 *: The optimization problem (6) in the case of $\mathcal{H}_2$-norm used can be solved by the following LMI optimization problem*

$$(RLH2) \min_{X,Q,Q_k,\nu} \quad \nu : \tag{16}$$

$$\mathcal{N}_{[C\quad D]}^T \begin{bmatrix} -X + A^TXA & A^TXB \\ B^TXA & -I + B^TXB \end{bmatrix} \mathcal{N}_{[C\quad D]} < 0 \tag{17}$$

$$\begin{bmatrix} X & L^T \\ L & \nu \end{bmatrix} > 0 \tag{18}$$

$$Q = E_{0k}^T Q_k E_{0k},\ Q_k \geq 0$$

$$\begin{bmatrix} -X+Q & 0 & A^T(X-Q) \\ 0 & -I & B^T(X-Q) \\ (X-Q)A & (X-Q)B & -X+Q \end{bmatrix} < 0 \tag{19}$$

where E_{0k} is the $k\times n$ matrix with k unit column vectors and $n-k$ zero column vectors. The optimal k-order filter (11) is easily derived from the optimal solution of (16) as follows: $\mathcal{X}$ is defined as the formula

$$\mathcal{X} = \begin{bmatrix} X & E_{0k} \\ E_{0k} & Q_K^{-1} \end{bmatrix}.$$

Then, we solve the following LMIs in respect to variables $K = [\,A_F \quad B_F\,]$ and L_F.

$$\begin{bmatrix} -\mathcal{X} & * & * \\ 0 & -I & * \\ \Theta_k A \Theta_k^T & B_{1,ak} & -\mathcal{X}^{-1} \end{bmatrix} + \left(\begin{bmatrix} 0 \\ 0 \\ B_{ak} \end{bmatrix} K \,[C_{ak} \quad D_{21,ak} \quad 0] + (*) \right) < 0, \tag{20}$$

$$\begin{bmatrix} \mathcal{X} & C_{1,ak}^T \\ C_{1,ak} & \nu \end{bmatrix} + \begin{bmatrix} 0 \\ D_{ak}^T \end{bmatrix} L_F^T \,[0 \quad -I] + \begin{bmatrix} 0 \\ -I \end{bmatrix} L_F \,[D_{ak} \quad 0] > 0\,. \tag{21}$$

in which $\mathcal{A}_{cl} = \Theta_k A \Theta_k^T + B_{ak} K C_{ak}$, $\mathcal{B}_{cl} = B_{1,ak} + B_{ak} K D_{21,ak}$, $C_{1,ak} = [\,L \quad 0_k\,]$, $D_{ak} = [\,0_{kn} \quad I_n\,]$,

$$\Theta_k = \begin{bmatrix} I_n \\ 0_{nk} \end{bmatrix}, B_{ak} = \begin{bmatrix} 0_{nk} \\ I_k \end{bmatrix}, C_{ak} = \begin{bmatrix} 0_{kn} & I_k \\ C & 0_{1k} \end{bmatrix}, \quad B_{1,ak} = \begin{bmatrix} B \\ 0_{k1} \end{bmatrix}, D_{21,ak} = \begin{bmatrix} 0_{k1} \\ D \end{bmatrix}.$$

3.2. Optimal $\mathcal{H}_\infty$ synthesis filter

The time-domain interpretation of $\mathcal{H}_\infty$-norm defined by (4) is the following input-output relation of system (15)

$$\sup_{\sum_{i=0}^{\infty} ||w_i||^2 = 1} \sqrt{\sum_{i=0}^{\infty} ||z_i||^2},$$

The optimization problem (6) with $\mathcal{H}_\infty$-norm used is solved by one of the equivalent optimization problems expressed briefly as the following:

Theorem 2 *The problem is solved by*

$$(RLH\infty) \quad \min_{X,Q,Q_k,\gamma} \gamma : \tag{22}$$

$$Q = E_{0k}^T Q_k E_{0k}, \; Q_k \geq 0$$

$$\begin{bmatrix} -X + Q & * & * \\ 0 & -\gamma I & * \\ (X - Q)A & (X - Q)B & -X + Q \end{bmatrix} < 0 \tag{23}$$

$$\begin{bmatrix} \mathcal{N}^T_{[C \quad D]} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1 & T_2 & L \\ T_3 & T_4 & 0 \\ L^T & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{N}_{[C \quad D]} & 0 \\ 0 & I \end{bmatrix} < 0, \tag{24}$$

where $T_1 = -X + A^T X A$, $T_2 = A^T X B$, $T_3 = B^T X A$, $T_4 = -\gamma I + B^T X B$.

Moreover, note that once a solution X, Q of (22) has been found, the optimal $k-$order filter (12) can be easily derived by a procedure similarly to that in the end of the previous subsection.

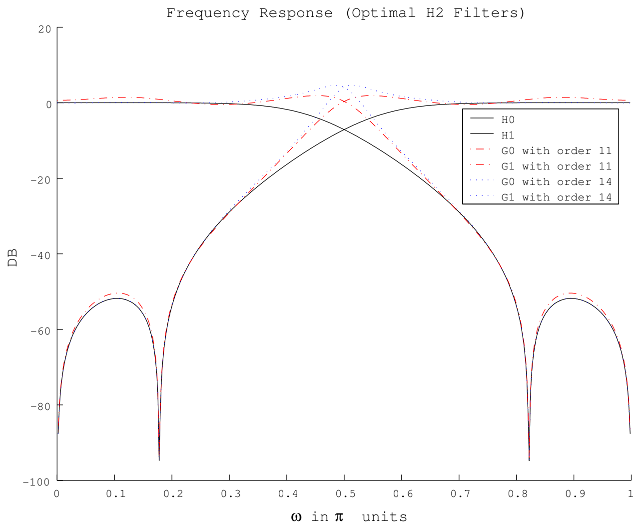

Fig. 1. Frequency response of analysis and synthesis filters, the results of $\mathcal{H}_2$ filters

4. FIXED-ORDER OPTIMAL MULTI-CRITERION FILTERS

In this subsection, we discuss the solution for the multi-criterion optimization problems (8), (9). In the discussion below, we now provide a direct approach, which require much large dimensional LMIs with more matrix-variables involved. The following result is based on LMI optimization formulation for problems (8), (9), where each $\mathcal{H}_\infty$-norm and $\mathcal{H}_2$-norm criterion is checked by its own Lyapunov variables $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$, respectively.

$$\min_{\hat{\mathcal{X}},\hat{\mathcal{Y}},\hat{V},\hat{K},\hat{L}_F,\gamma,\nu} \mu\gamma + (1-\mu)\nu : \tag{25}$$

$$\begin{bmatrix} -\hat{\mathcal{X}} & * & * \\ 0 & -I & * \\ T_{11} & T_{12} & T_{13} \end{bmatrix} < 0, \tag{26}$$

$$\begin{bmatrix} \hat{\mathcal{X}} & * \\ [\,L \quad -\hat{L}_F\,] & Z \end{bmatrix} > 0, \; Z < \nu \tag{27}$$

$$\begin{bmatrix} -\hat{\mathcal{Y}} & * & * & * \\ 0 & -\gamma I & * & * \\ T_{15} & T_{16} & T_{17} & * \\ T_{18} & -M_F D & 0 & -\gamma I \end{bmatrix} < 0, \tag{28}$$

in which

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \Pi_V = \begin{bmatrix} I_n & 0 \\ 0 & \tilde{V}_{12} V_{22}^{-1} \end{bmatrix}, V_{11} \in R^{n\times n},$$

$$V_{12} = E_{0k}^T \tilde{V}_{12}, \; \tilde{V}_{12} \in R^{k\times k}, \hat{\mathcal{X}} = \Pi_V \mathcal{X} \Pi_V^T,$$

$$\hat{K} = \tilde{V}_{12}\,[\,A_F V_{22}^{-T} \tilde{V}_{12}^T \quad B_F\,], \Lambda_k = \begin{bmatrix} E_{0k}^T \\ I_k \end{bmatrix},$$

$$\hat{V} = \Pi_V V \Pi_V^T = \begin{bmatrix} V_{11} & E_{0k}^T \tilde{V}_{12} V_{22}^{-T} \tilde{V}_{12}^T \\ \tilde{V}_{12} V_{22}^{-1} V_{21} & \tilde{V}_{12} V_{22}^{-T} \tilde{V}_{12}^T \end{bmatrix},$$

$$\hat{V}_1 \in R^{n\times n},\ \hat{V}_2 \in R^{k\times k},\ \hat{V}_3 \in R^{n\times k},$$

$T_{11} = \hat{V}\Theta_k A\Theta_k + \Lambda_k \hat{K} C_{ak}$, $T_{12} = \hat{V}\Theta_k B + \Lambda_k \hat{K} D_{21,ak}$, $T_{13} = \hat{\mathcal{X}} - (\hat{V} + \hat{V}^T)$, $\hat{L}_F = L_F(\tilde{V}_{12} V_{22}^{-1})^T$, $T_{15} = \hat{V}\Theta_k A\Theta_k + \Lambda_k \hat{K} C_{ak}$, $T_{16} = \hat{V}\Theta_k B + \Lambda_k \hat{K} D_{21,ak}$, $T_{17} = \hat{\mathcal{Y}} - (\hat{V} + \hat{V}^T)$, $T_{18} = [\, L - M_F C \quad -\hat{L}_F \,]$. Once solution (25) is found, $K = [\, A_F \quad B_F \,]$ and L_F defining the filter G in (11) are derived by the following formula

$$A_F = \hat{A}_F \hat{V}_2^{-1},\ B_F = \hat{B}_F,\ L_F = \hat{L}_F \hat{V}_2^{-1}. \qquad (29)$$

5. SIMULATION

We use the same low-pass third-order IIR Chebyshev filter $H_0(z)$ as in [1]

$$H_0(z) = \frac{0.1412 + 0.3805z^{-1} + 0.3805z^{-2} + 0.1412z^{-3}}{1 - 0.3011z^{-1} + 0.3694z^{-2} - 0.0250z^{-3}}$$

and $H_1(z) = H_0(-z)$, which is high-pass. Since such best error performance becomes extremely small at delay $n_o = 10$, we take this value of n_0 for further design of low-order synthesis filters $G(z)$ in (6), whose size varies from 8 to 11, so the order of synthesis filters, G_0 and G_1 in (1)-(2) accordingly varies from 11 to 14. For filter order comparison, note that $\mathcal{H}_\infty$ control based formulation of [1] results in order 34 of their designed filters. The values of matrix E_{ok} is generated randomly in many trials for $\mathcal{H}_2$ and $\mathcal{H}_\infty$-filters to select its best solution. All the synthesis filters G_0, G_1 provided below are also low-pass and high-pass, respectively. Figure 1 shows frequency responses of two analysis filters H_0, H_1 and those of two synthesis filters G_0, G_1, generated by $\mathcal{H}_2$ filter design in two cases of 11 and 14 order. Two synthesis filters G_0, G_1, which are the results of $\mathcal{H}_\infty$ and mixed-norm filter design, have the same shapes to the results of $\mathcal{H}_2$ filter design with a little difference in magnitude distortion. In figure 1, the 14-th order synthesis filters track both stopband and passband of analysis filters and its result is better than 11-order filters does. In figure 2, as expected, the distortion with $\mathcal{H}_2$ filters has peak value larger than that with $\mathcal{H}_\infty$ and mixed-norm ones, but its energy distortion is smaller than that of the others. The peak distortion (energy distortion, resp.) with the mixed filter is larger (smaller, resp.) than that with $\mathcal{H}_\infty$ filter but still smaller (larger, resp.) than that with $\mathcal{H}_2$-filter.

6. CONCLUSIONS

In constrast to FIR filter design theory, which is mature enough, there are a lot of open issues to be addressed in IIR filters. In this paper we have propsed an effective method for solving the low-order IIR QMF bank design, which is certainly among most challenging issues in the filter bank design. Three optimization criteria were discussed: $\mathcal{H}_2$-filter, $\mathcal{H}_\infty$ filter and mixed-norm filters. Our proposed method yields IIR synthesis filters of much smaller order comparing to existing methods thus reduces the implementation complexity of the overall filter bank.

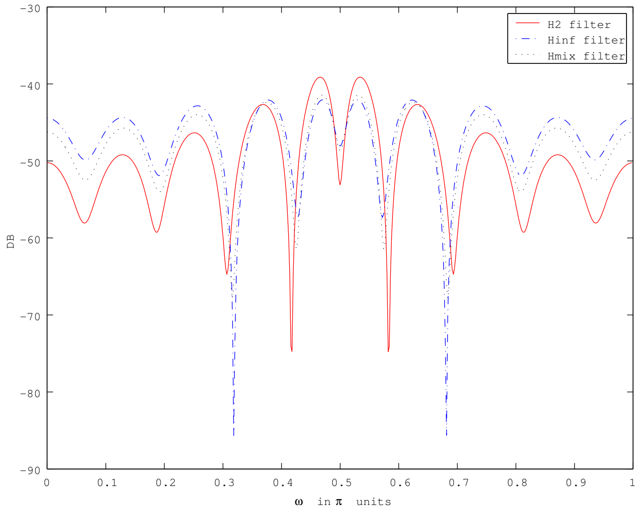

Fig. 2. Magnitude of frequency error $|z^{-n_0} - (H_0(z)G_0(z) + H_1(z)G_1(z))|$ with 14-order $\mathcal{H}_2$, $\mathcal{H}_\infty$ and mixed-norm synthesis filters

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