A DESIGN AND IMPLEMENTATION OF ORTHONORMAL SYMMETRIC WAVELET TRANSFORM USING PRCC FILTER BANKS

Abhijin Adiga *K.R. Ramakrishnan*

Department of Electrical Engineering,
Indian Institute of Science,
Bangalore, India

B.S.Adiga

Philips Innovation Campus,
Bangalore, India

ABSTRACT

We propose a framework for orthonormal symmetric wavelet transform given a cyclic zero-phase half band filter. The scheme consists of a design for cyclic wavelet transform with real symmetric filters based on the Perfect Reconstruction Circular Convolution Filter Banks accompanied by an implementation in the Discrete Trigonometric Transform domain. This is followed by a brief discussion on its performance in the context of applications such as image compression and implementation of symmetric wavelet transforms based on bandlimited wavelets.

1. INTRODUCTION

In this paper, we propose a design for Perfect Reconstruction Circular Convolution Filter Banks (*PRCC-FB*) [1] which yields real symmetric-periodic analysis and synthesis filters. This is accompanied by an implementation in the Discrete Trigonometric Transform (DTT) domain under the assumption of symmetric input and output sequences.

The proposed framework can be used to implement Symmetric Wavelet Transforms (*SWT*) [2]. *SWT* is a discrete wavelet transform with symmetric inputs and outputs, and requires the use of linear-phase filters. It solves the problems of coefficient expansion and border discontinuities inherent in cyclic wavelet transforms.

The paper is organized as follows. In section-2 *PRCC-FB* are briefly described. In section-3 the required types of symmetry and associated *DTTs* have been discussed. In section-4 we describe the proposed design, and provide some examples to illustrate the scheme. It is followed by the implementation in section-5. In section-6 certain issues pertaining to the use of this framework in implementation of bandlimited wavelet transforms and image compression are discussed.

2. PRCC FILTER BANKS

PRCC-FB is a two-band cyclic filter bank which satisfies the *cyclic paraunitary property* [3] and hence corresponds to an orthonormal transformation. It is designed and implemented entirely in the *DFT* domain. The implementation is illustrated in fig.1, where $x(n)$ is the input sequence of length N; $H_0(k)$ and $H_1(k)$ are the *analysis filters*; $F_0(k)$ and $F_1(k)$ are the *synthesis filters*; $W_0(k)$ and $W_1(k)$ are the subband sequences. The general design of PRCC-FB follows.

The construction of filters described here has been discussed in detail in [1],[4]. $H_0(k)$ must satisfy the *power complementary* condition following which it takes the general form,

$$H_0(k) = |H_0(k)|e^{j\phi(k)} = H_{hb}(k)^{1/2}e^{j\phi(k)} \tag{1}$$

where, $H_{hb}(k)$ is a *zero-phase half band filter* [5] and $\phi(k)$ is the *phase term*. $H_{hb}(k)$ must satisfy the following properties,

1. $H_{hb}(N-k) = H_{hb}(k)$
2. $H_{hb}(N/2-k) = 1 - H_{hb}(k)$
3. $H_{hb}(N/2+k) = H_{hb}(N/2-k)$

where, all the indices are $\pmod N$. N is assumed to be even throughout. The remaining filters are derived as follows,

$$\begin{aligned} H_1(k) &= -e^{j2\pi k/N}H_0^*(N/2+k) \\ F_0(k) &= e^{j2\pi k/N}H_0^*(k) \\ F_1(k) &= e^{j2\pi k/N}H_1^*(k) \end{aligned} \tag{2}$$

It is to be noted that the conditions for perfect reconstruction do not depend on $\phi(k)$.

3. SYMMETRIC EXTENSIONS OF SIGNALS AND ASSOCIATED DTTS

In this section we describe two types of extensions and their relationships with certain *DTTs*. A thorough investigation has been carried out in [6] regarding the symmetric extension of sequences, their association with the family of *DTTs* and *symmetric convolution*.

Consider a *finite-length real* sequence $x(n)$ of length N (even). Its *type 2 DCT* is as follows,

$$X^{c2}(k) = 2\sum_{n=0}^{N-1} x(n)\cos\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right) \tag{3}$$

for $0 \le k \le N-1$. Let $x_s(n)$ be the *symmetric-periodic* extension of $x(n)$, a period of which is defined as

$$x_s(n) = x(n) + x(2N-n-1) \tag{4}$$

for $0 \le n \le 2N-1$. Referring to [6],[7], $x_s(n)$ is a *HSHS Symmetric Periodic sequence* (HSHS-SPS) where, HS stands for *Half-sample Symmetry*. The *DFT* of $x_s(n)$ can be expressed as,

$$X_s(k) = e^{j\pi k/(2N)}X^{c2}(k) \quad 0 \le k \le 2N-1 \tag{5}$$

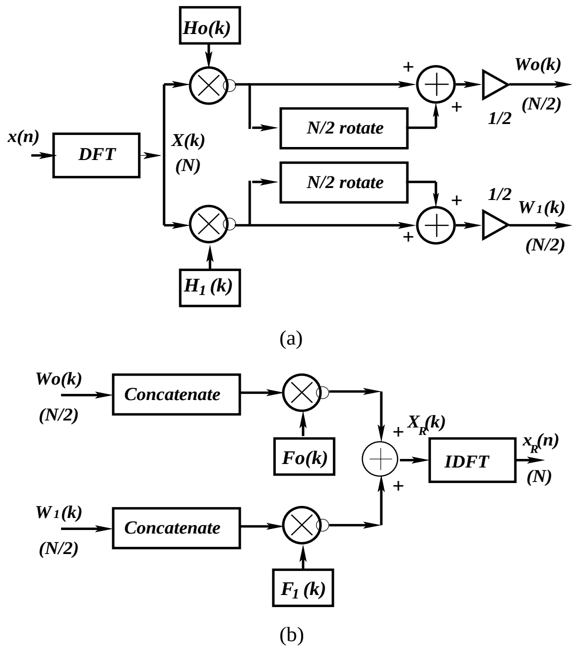

Fig. 1. PRCC filter bank system. (a) Analysis filter bank (b) Synthesis filter bank. (courtesy:[1])

Using (3) it can be shown that,

$$\begin{array}{rcll} X^{c2}(2N-k) & = & -X^{c2}(k), & 0 \le k \le 2N-1, \\ X^{c2}(4N-k) & = & X^{c2}(k), & 0 \le k \le 4N-1 \end{array} \quad (6)$$

It is easily seen that substituting any real sequence of the form (6) in (5) corresponds to the *DFT* of a *HSHS-SPS*.

Similarly, the *type 2 Discrete Sine Transform* (DST) of $x(n)$ is,

$$X^{s2}(k) = 2\sum_{n=0}^{N-1} x(n)\sin\left(\frac{\pi k\left(n+\frac{1}{2}\right)}{N}\right) \quad 1 \le k \le N \quad (7)$$

Let $x_a(n)$ be the *antisymmetric periodic* extension of $x(n)$,

$$x_s(n) = x(n) - x(2N-n-1) \quad (8)$$

for $0 \le n \le 2N-1$. $x_a(n)$ is a *HAHA-SPS* (HA stands for *Half-sample Antisymmetry*). The *DFT* of $x_a(n)$ can be expressed as

$$X_a(k) = -je^{j\pi k/(2N)} X^{s2}(k) \quad 0 \le k \le 2N-1 \quad (9)$$

Using (7),

$$\begin{array}{rcll} X^{s2}(k) & = & X^{s2}(2N-k), & 0 \le k \le 2N-1, \\ X^{s2}(4N-k) & = & -X^{s2}(k), & 0 \le k \le 4N-1 \end{array} \quad (10)$$

where, $X^{s2}(k) = 0$ for $k = 0$. Again, it can be observed that substituting any real sequence of the form (10) in (9) corresponds to the *DFT* of a *HAHA-SPS*. Note that the expressions for *DCT* and *DST* are not normalized. All the results in this paper can be easily normalized by multiplying with appropriate scaling factors.

4. PROPOSED FILTER DESIGN

In [1], symmetric extension of input signal was considered to overcome the edge artifacts problem inherent in *PRCC-FB* due to its cyclic nature. There, the *zero-phase symmetry* of the filters was utilized. Here, we propose a different symmetric extension.

The phase term $\phi(k)$ in (1) is defined such that,

$$H_0(k) = \begin{cases} H_{hb}(k)^{1/2} e^{j\pi k/(2N)} & 0 \le k \le N-1 \\ -H_{hb}(k)^{1/2} e^{j\pi k/(2N)} & N \le k \le 2N-1 \end{cases} \quad (11)$$

where the length of the filter is $2N$. Using (5) and (6), it can be easily shown that the above design renders the *IDFT* of $H_0(k)$ *HSHS-SPS*. Using (2) the remaining filters are obtained. The *IDFT* of $H_1(k)$ corresponds to a *HAHA-SPS*. The synthesis filters are related to the analysis filters as follows,

$$\begin{array}{rcl} F_0(k) & = & H_0(k) \\ F_1(k) & = & -H_1(k) \end{array} \quad (12)$$

which implies that the *IDFTs* of $F_0(k)$ and $F_1(k)$ are *HSHS-SPS* and *HAHA-SPS* respectively.

Let H_0^{c2} and F_0^{c2} be the *DCTs of the first half of the IDFTs* of H_0 and F_0 respectively. In the same way, let H_1^{s2} and F_1^{s2} be the *DSTs* associated with H_1 and F_1. From (2),(5) and (9), we can derive the following,

$$H_1^{s2}(k) = H_0^{c2}(N-k) \quad (13)$$

for $0 \le k < N$. Equation (13) implies that H_1^{s2} can be obtained by flipping H_0^{c2}. We provide two designs, one based on the *IIR* filter banks and the other based on the popular Daubechies filter set, to exemplify the above scheme.

Example 4.1: The power spectrum of a Meyer scaling function satisfies the conditions of being a zero-phase half band filter. An orthonormal symmetric filter bank based on the generalized Meyer class of bandlimited wavelets [1],[4],[8] can be constructed as follows. Firstly, the power spectrum of the low pass analysis filter must be appropriately sampled [1] to obtain H_{hb}. Then the design technique described above is applied to obtain the symmetric-periodic filters. The low pass analysis filter for $2N = 128$ thus derived is plotted in fig.2.

Example 4.2: Consider the Daubechies 8 coefficient filter set and choose $N \ge 4$. The power spectrums of the filters correspond to that of zero-phase half band filters. First, a $2N$-point *DFT* of the FIR low pass analysis filter h_0^{or} (“or” corresponds to the original set), say H_0^{or} is computed. Then, $H_{hb} = |H_0^{or}k|^2$, the power spectrum of h_0^{or} at $2N$-point resolution. Finally, our design technique is employed. The low pass analysis filter for $2N = 128$ is plotted in fig.2.

The same strategy can be applied to the half band filter designed in [9]. The filter sets for a multilevel decomposition can be derived by decimation of the preceding filter set in the *DFT* or *DTT* domain.

5. A DTT DOMAIN IMPLEMENTATION

The design described in section-4 can be implemented using type-2 *DCTs* and type-2 *DSTs* of the sequences involved. In [10], *DTTs*

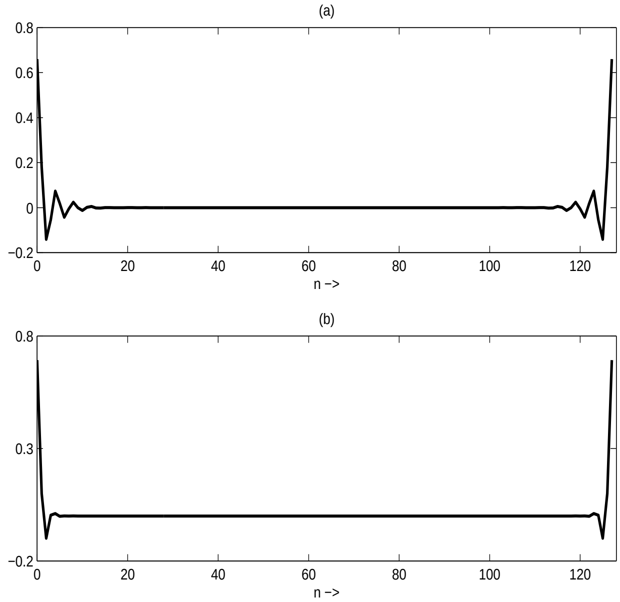

Fig. 2. Symmetric-periodic low pass analysis filters derived from (a) Meyer half band filter (b) Daubechies (coeff-8) half band filter

have been used, but only for the filtering operations. Here, we utilize the periodicity of the filters to perform even decimation and interpolation operations in the *DTT* domain itself (i.e. on the lines of [1]).

The filter bank implementation is divided into four parts. For each part we derive the equivalent *DTT* domain implementation.

5.1. Low Pass Filtering and Decimation

Let $x(n)$ be the real input sequence of length N (even). It is extended as in (4) and filtered by $H_0(k)$ of section-4. Let $U_0(k)$ be the resulting $2N$ point sequence. Using (5),

$$U_0(k) = e^{j\pi k/(2N)} X^{c2}(k) \times e^{j\pi k/(2N)} H_0^{c2}(k) \tag{14}$$

for $0 \leq k \leq 2N-1$. The decimation can be implemented using *DFT* domain techniques described in [1], [3]. Let $W_0(k)$ be the *DFT* of the decimated sequence.

$$W_0(k) = \left[U_0(k) + U_0(N+k)\right]/2 \tag{15}$$

Substituting (14) in (15) and using (6),

$$W_0(k) = \frac{e^{j\pi k/N}}{2} \underbrace{\left[X^{c2}(k)H_0^{c2}(k) - X^{c2}(N-k)H_0^{c2}(N-k)\right]}_{S_0(k)} \tag{16}$$

for $0 \leq k \leq N-1$. The term $S_0(k)$ in (16) has the form (6). This implies $IDFT(W_0(k))$ is a *HSHS-SPS*. Let $W_0^{c2}(k)$ be the *DCT* of the first half of $IDFT(W_0(k))$. Then,

$$W_0^{c2}(k) = \frac{1}{4}\left\{X^{c2}(k)H_0^{c2}(k) - X^{c2}(N-k)H_0^{c2}(N-k)\right\} \tag{17}$$

for $0 \leq k \leq N/2-1$. It is to be noted that $X^{c2}(N) = H_0^{c2}(N) = 0$. From the above equation we see that low pass filtering followed by decimation can be accomplished in the *DCT* domain itself.

5.2. High Pass Filtering and Decimation

We proceed the same way as in the previous section using H_1^{s2} and (9) instead. The resulting subband sequence $W_1(k)$ is as follows,

$$W_1(k) = -j\frac{e^{j\pi k/N}}{2} \underbrace{\left[X^{c2}(k)H_1^{s2}(k) + X^{c2}(N-k)H_1^{s2}(N-k)\right]}_{S_1(k)} \tag{18}$$

for $0 \leq k \leq N-1$, The term $S_1(k)$ of (18) is of the form (10) and therefore, (18) is similar to (9). This implies $IDFT(W_1(k))$ is a *HAHA-SPS*. Let $W_1^{s2}(k)$ be the *DST* of the first half of $IDFT(W_1(k))$. Then,

$$W_1^{s2}(k) = \frac{1}{4}\left[X^{c2}(k)H_1^{s2}(k) + X^{c2}(N-k)H_1^{s2}(N-k)\right] \tag{19}$$

for $1 \leq k \leq N/2$. Thus, the high pass filtering followed by decimation operation is accomplished in the *DTT* domain.

5.3. Upsampling and Low Pass Filtering

Here, $W_0(k)$ (the output of low pass filtering and decimation) is the input sequence which can be expressed as,

$$W_0(k) = e^{j\pi k/N} W_0^{c2}(k) \qquad 0 \leq k \leq N-1 \tag{20}$$

Upsampling can be implemented as described in [1],[3]. The $2N$-point *DFT* of the interpolated sequence is,

$$W_i(k) = W_0(k) \quad 0 \leq k \leq 2N-1 \tag{21}$$

This sequence is filtered by $F_0(k)$. Let the resulting sequence be $Y_0(n)$. Employing (5) for $F_0(k)$ and rearranging,

$$Y_0(k) = e^{j\pi k/N}\left[e^{j\pi k/(2N)} W_0^{c2}(k) F_0^{c2}(k)\right] \tag{22}$$

for $0 \leq k \leq 2N-1$. The term $e^{j\pi k/N}$ in (22) corresponds to a *circular left shift* of the *IDFT* of the term in the braces by 1 sample. Now we consider the filter bank framework described in section-4. The phase condition imposed on the filters corresponds to a *circular left shift* of the reconstructed sequence by 1 sample relative to the original sequence. Therefore, the resulting sequence needs to be shifted circularly to the right by 1 sample. Instead we ignore the term $e^{j\pi k/N}$ in (22),

$$\overline{Y}_0(k) = e^{j\pi k/(2N)} \underbrace{\left[W_0^{c2}(k)F_0^{c2}(k)\right]}_{S_2(k)} \tag{23}$$

for $0 \leq k \leq 2N-1$, where, $\overline{Y}_0(k) = e^{-j\pi k/N} Y_0(k)$. The term $S_2(k)$ is similar to (6) which implies that (23) is of the form (5). Let $\overline{Y}_0^{c2}(k)$ be the *DCT* of the first half of the *IDFT* of $\overline{Y}_0(k)$.

$$\overline{Y}_0^{c2}(k) = \begin{cases} \frac{1}{2}W_0^{c2}(k)F_0^{c2}(k), & 0 \leq k \leq \frac{N}{2}-1, \\ 0, & k = \frac{N}{2}, \\ -\frac{1}{2}W_0^{c2}(N-k)F_0^{c2}(k), & \frac{N}{2}+1 \leq k \leq N-1 \end{cases} \tag{24}$$

Thus we see that the operations of upsampling followed by low pass filtering can be performed using the *DCTs* of the sequences involved.

5.4. Upsampling and High Pass Filtering

Adopting the same procedure as above, but using 9 and $F_1(k)$ instead, the sequence $\overline{Y}_1(k)$ is derived.

$$\overline{Y}_1(k) = -e^{j\pi k/(2N)} \underbrace{\left[W_1^{s2}(k)F_1^{s2}(k)\right]}_{S_3(k)} \quad (25)$$

for $0 \leq k \leq 2N-1$. $S_3(k)$ is similar to (6). The equation (25) is of the form (5). Let $\overline{Y}_1^{c2}(k)$ be the *DCT* of the first half of the *IDFT* of $\overline{Y}_1(k)$.

$$\overline{Y}_1^{c2}(k) = \begin{cases} 0, & k=0, \\ -\frac{1}{2}W_1^{s2}(k)F_1^{s2}(k), & 1 \leq k \leq \frac{N}{2}, \\ -\frac{1}{2}W_1^{s2}(N-k)F_1^{s2}(k), & \frac{N}{2}+1 \leq k \leq N-1 \end{cases} \quad (26)$$

From the above equation it is seen that upsampling followed by high pass filtering can be implemented completely in the *DTT* domain.

$X^{c2}(k)$, the *DCT* of the input sequence can be reconstructed by, $X^{c2}(k) = \overline{Y}_0^{c2}(k) + \overline{Y}_1^{c2}(k)$. Hence, the filter bank of section-4 can be implemented completely in the DTT domain using (17),(19),(24) and (26).

6. COMPUTATIONAL ANALYSIS AND POSSIBLE APPLICATIONS

The *DTT* domain implementation of a one level 1-D wavelet decompostion requires the computation of 1 N-point *DCT*, 2 Hadamard products of real N-length vectors, 1 $N/2$-point *IDCT*, and 1 $N/2$-point *IDST*. The computation of 1-D N-point *DCT* or *DST* requires $\left(\frac{N}{2}\log N\right)$ multiplications and $\left(\frac{3N}{2}\log N - N + 1\right)$ additions [11]. The total number of multiplications for a one level decomposition is $N(\log N + 1.5)$ as against $2N(\log N + 0.5)$ for the *FFT* domain implementation of the same [12]. The total number of additions is $\left(3N\log N - \frac{5}{2}N + 3\right)$.

It is evident that the proposed filter bank structure is well suited for implementing *SWT* based on orthogonal bandlimited wavelets. However, it should be noted that the application of the proposed structure is not limited to bandlimited wavelets alone.

We applied the proposed scheme on images using filter sets of examples 4.1 and 4.2 and compared the performance with that of the popular $9/7$ biorthogonal filter set. The results are given in table 1. It is to be noted that after the decomposition is performed using (17) and (19), the results in the *DTT* domain can be quantized and encoded directly rather than perform the *IDCTs* and *IDSTs* of the individual subbands. Also, a multilevel decomposition can be achieved by computing only the 2 Hadamard products for each level after the *DCT* of the input signal is initially computed. This way the complexity of the scheme can be greatly reduced.

7. CONCLUSION

We proposed a design and implementation scheme for a class of orthonormal cyclic wavelet transforms using PRCC filter banks. The design yields a real and symmetric filter set given a half band filter. To supplement it, a *DTT* domain implementation which assumes symmetric input and output sequences was proposed. The framework may serve as an efficient implementation of *SWT* based on bandlimited wavelets and may also find application in image compression.

Method		*avg. PSNR*
Our scheme based on	Meyer	24.9525
	Daub. 8	24.9395
	Daub. 32	24.9524
Biorth. 9/7		24.6481

Table 1. Average PSNR in db for 12 test images [13] using a two level transform and reconstructing only the low pass component.

8. REFERENCES

[1] Ajit S. Bopardikar, M. R. Raghuveer, and B. Suryanarayana Adiga, "Perfect reconstruction circular convolution filter banks and their application to the implementation of bandlimited discrete wavelet transform," *IEEE Int. Conf. Acoust., Speech, Signal Process.*, pp. 3665–3668, Apr. 1997.

[2] Bryan E. Usevitch, "A tutorial on modern lossy wavelet image compression: Foundations of jpeg 2000," *IEEE Signal Processing Magazine*, vol. 18, pp. 22–35, Sept. 2001.

[3] P. P. Vaidyanathan and Ahmet Kirac, "Cyclic LTI systems in digital signal processing," *IEEE Trans. on Signal Processing*, vol. 47, no. 2, pp. 433–447, Feb. 1999.

[4] Raghuveer M. Rao and Ajit S. Bopardikar, *Wavelet Transforms, Introduction to Theory and Applications*, Addison-Wesley, 1998.

[5] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall P T R, Englewood Cliffs, NJ, 1993.

[6] Stephen A. Martucci, "Symmetric convolution and the discrete sine and cosine transforms," *IEEE Trans. on Signal Processing*, vol. 42, no. 5, pp. 1038–1051, May 1994.

[7] Gilbert Strang and Truong Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, 1996.

[8] Stéphane Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, 1998.

[9] R. Motwani and K. R. Ramakrishnan, "Design of two-channel linear phase orthogonal cyclic filter banks," *IEEE Signal Processing Lett.*, vol. 5, pp. 121–123, May 1998.

[10] Stephen A. Martucci and R. M. Mersereau, "The symmetric convolution approach to the nonexpansive implementation of fir filter banks for images," *IEEE Int. Conf. Acoust., Speech, Signal Process.*, pp. V.65–V.68, Apr. 1993.

[11] P. Lee and F. Huang, "Restructured recursive DCT and DST algorithms," *IEEE Trans. on Signal Processing*, vol. 42, no. 7, pp. 1600–1609, July 1994.

[12] M. Ramkumar, G. V. Anand, and Ali N. Akansu, "The lifting scheme: A custom-design construction of biorthogonal wavelets," *NSF industry/university co-operative research centre for digital video and media, Doc. no. CDMVR TR-99-13*, Apr. 1999.

[13] "Waterloo bragzone," *http://links.uwaterloo.ca/bragzone.base.html*.