NORM INDUCED QMF BANKS DESIGN USING LMI CONSTRAINTS

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ABSTRACT

This paper presents constrained ℓ_2 norm and L_∞ norm optimal QMF banks designs. Both methods cast the design problems as a linear objective function minimization problem subject to Linear Matrix Inequality (LMI) constraints, which are solved by semi-definite programming. The LMI constraints are shown to be convex. Consequently, the designed QMF banks are *globally* optimal with respect to the objective function.

1. INTRODUCTION

A two channel maximally decimated filter bank is shown in Fig.1, where $H_0(z)$, $H_1(z)$ and $F_0(z)$, $F_1(z)$ are the lowpass and highpass analysis subband filters, and lowpass and highpass synthesis subband filters respectively. The reconstructed signal $\hat{x}(n)$ is related to the input signal $x(n)$ as

$$\begin{aligned}\hat{X}(z) &= 0.5(H_0(z)F_0(z) + H_1(z)F_1(z))X(z) \\ &\quad +0.5(H_0(-z)F_0(z) + H_1(-z)F_1(z))X(-z) \\ &= T(z)X(z) + A(z)X(-z), \end{aligned} \quad (1)$$

where $T(z) = 0.5(H_0(z)F_0(z)+H_1(z)F_1(z))$ is the linear transfer function of the filter bank, and $A(z) = 0.5(H_0(-z)F_0(z) + H_1(-z)F_1(z))$ is the aliasing component of the filter bank. The filter bank is Perfect Reconstruction (PR) when $A(z) = 0$, and $T(z) = cz^{-\ell}$ for some nonzero $c \in \mathbb{R}$ and $\ell \in \mathbb{Z}$, such that the overall filter bank is a scaled delay system.

Traditionally, the analysis and synthesis filters are Quadrature Mirror (QM) related,

$$H_1(z) = H_0(-z), \quad F_0(z) = 2H_1(-z), \quad \text{and} \quad F_1(z) = -2H_0(-z). \quad (2)$$

When the subband filters are QM related, the aliasing component is structurally nullified, i.e. $A(z) = 0$. As a result, the PR Quadrature Mirror Filter (QMF) bank design problem can be formulated as an optimization problem of $T(z)$ towards the desired transfer function, i.e. a scaled delayed function $cz^{-\ell}$, with $H_0(z)$ as variable. Without loss of generality, assume $c = 1$, therefore, the PR constraint is expressed as

$$T(z) = H_0^2(z) - H_0^2(-z) = z^{-\ell}. \quad (3)$$

It's observed that the PR constraint in eq.(3) can be replaced by the equivalent power complementary constraint as

$$1/\alpha \leq | H_0(e^{j\omega}) |^2 + | H_0(e^{j(\pi-\omega)}) |^2 \leq \alpha, \quad \omega \in [0, \frac{\pi}{2}], \quad (4)$$

where α specifies the biggest allowable ripple size of the reconstruction error. Consider the PR constraint in eq.(4), the minimization objective function achieved by Weighted Least Square (WLS) optimal QMF banks can be formulated as

$$\begin{aligned}\min_{h_0(n)} E &= \sum_{\omega=0}^{\frac{\pi}{2}} W(\omega)(| H_0(e^{j\omega}) |^2 + | H_0(e^{j(\omega+\pi)}) |^2 - 1)^2 \\ &\quad + \sum_{\omega=\omega_s}^{\pi} W(\omega) \, | H_0(e^{j\omega}) |^2, \end{aligned} \quad (5)$$

where $W(\omega)$ is the spectral weighting function. The first term denotes the weighted spectral energy of the reconstruction error, and the second term denotes the weighted spectral energy of the subband filter stopband such that the resulting subband filters have good spectral separations. The L_2 optimal WLS QMF banks design problem in eq.(5) can be recast as a constrained optimization problem [3, 4]

$$\min_d \quad E = d^T\mathbf{R}d, \qquad \text{Subject to} \quad d^T\mathbf{B}d = 1, \quad (6)$$

where d specifies the impulse response of $H_0(z)$, **R** and **B** are positive definite matrices. **R** can be the cosine matrix that computes the spectral energy E in eq.(5). If **B** is an identity matrix, the constraint in eq.(6) will simply normalize the power of $H_0(z)$ such as to avoid trivial solution in the minimization problem. The optimization program in eq.(6) is a *generalized eigenfilter* design problem. The minimal value of E equals to the minimal generalized eigenvalue λ_{min} of (**R**, **B**) with d being the corresponding eigenvector [5]. However, **R** in the generalized eigenfilter problem is either singular or has small determinant. As a result, it is numerically unstable to determine the generalized eigenvalue of (**R**, **B**). Besides, it is difficult to integrate additional constraints that are imposed on the QMF banks. An alternative method to solve eq.(5) is to cast E into quadratic function of the filter coefficients [1], [2]. However, nonlinear optimization is required, which may lead to locally optimal solution and slow convergence.

A novel method is presented in this paper to design the QMF banks that is equivalent to the generalization of WLS design method. The proposed optimal design minimize the ℓ_2 norm of the lowpass analysis subband filter, which is related to the L_2 norm of the same filter by Parseval's theorem. The L_2 norm optimal QMF banks can be interpreted to be Constrained Least Square (CLS) optimal [6]. Furthermore, the design problem was cast as the linear objective minimization with Linear Matrix Inequality (LMI), which are solved by the readily available semi-definite programming tools [7]. The LMI constraints are linear and convex, which leads to global optimality.

In a similar setting, a constrained minimax design method is presented, which results in subband filters with equi-ripple magnitude response. The design of the subband filters with equi-ripple stopband magnitude response has been formulated as constrained nonlinear optimization problem in [1], and an iterative unconstrained nonlinear optimization algorithm is applied. The constrained nonlinear optimization problem is converted to an unconstrained WLS design problem, which was solved by an iterative algorithm [8]. Both design methods [1],[8] involve nonlinear optimization. The proposed design method avoided the nonlinear optimization problem. Via a *change of variable*, the equi-ripple QMF bank design problem is recast as linear optimization problem with LMI constraints, which are convex.

Linear Optimization subjects to LMI constraints has been applied to design digital filters in [9], such that the design problem can be solved by semi-definite programming. However, the design method presented in [9] cannot be used to design PR QMF bank due to the fact that PR is a nonlinear constraint with respect to the spectral response of individual subband filters, and hence to the filter coefficients. The proposed design method converted the PR constraint into LMI constraint in terms of the optimization variable such that semi-definite programming can be employed to optimize the magnitude response of individual subband filters and that of the overall QMF banks.

2. DESIGN PROBLEM FORMULATION

The QMF banks design can be formulated as a lowpass filter design problem subject to the PR constraint as in eq.(4). The design

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problem can be formulated to minimize the maximal stopband ripple size, $\sqrt{\tilde{\delta}}$, of the filter $H_0(z)$.

$$\text{Minimize} \quad \tilde{\delta} \tag{7}$$

$$\text{Subject to} \quad 1/\alpha \leq | H_0(e^{j\omega}) |^2 + | H_0(e^{j(\pi-\omega)}) |^2 \leq \alpha, \quad \omega \in [0, \tfrac{\pi}{2}], \tag{8}$$

$$| H_0(e^{j\omega}) |^2 \leq \tilde{\delta}, \quad \omega \in [\omega_s, \pi], \tag{9}$$

$$| H_0(e^{j\omega}) |^2 \geq 0, \quad \omega \in [0, \pi], \tag{10}$$

where the constraint in eq.(10) imposes $| H_0(e^{j\omega}) |$ being a proper magnitude response of $H_0(z)$. ω_s is the stopband edge of the lowpass filter. The above optimization program minimizes the biggest stopband ripple size within a given allowable maximal ripple size of the reconstruction error, which is equivalent to a constrained L_∞ optimal design of the stopband attenuation. Minimizing the biggest ripple size of the stopband results in equi-ripple stopband. That means, the "worst" energy leakage of the stopband is minimized when the input signal is separated by the subband filters into individual spectral domain. Alternatively, a PR constrained lowpass filter can be designed by minimizing the biggest allowable ripple size of the reconstruction error with a given maximal stopband ripple size $\sqrt{\delta}$.

$$\text{Minimize} \quad \tilde{\alpha} \tag{11}$$

$$\text{Subject to} \quad 1/\tilde{\alpha} \leq | H_0(e^{j\omega}) |^2 + | H_0(e^{j(\pi-\omega)}) |^2 \leq \tilde{\alpha}, \quad \omega \in [0, \tfrac{\pi}{2}], \tag{12}$$

$$| H_0(e^{j\omega}) |^2 \leq \delta, \quad \omega \in [\omega_s, \pi], \tag{13}$$

$$| H_0(e^{j\omega}) |^2 \geq 0, \quad \omega \in [0, \pi]. \tag{14}$$

The above optimization is equivalent to constrained L_∞ optimal design, where the biggest ripple size, which appears on the spectral response of the overall QMF bank, is minimized with the stopband ripple size of individual subband filters being equal or lower than the prescribed size $\sqrt{\delta}$. The "worst" case gain of the overall filter banks is minimized such that the system has excellent robustness with respect to noise disturbance.

3. LMI CONSTRAINTS FORMULATION

The constraints in the optimization problems of eq.(7) and eq.(11) are nonlinear with respect to $| H(e^{j\omega}) |$, which requires the application of nonlinear optimization. If the linear phase constraint is imposed on the FIR filter $H_0(z)$, the constraints in eq.(8-10) can be converted into *nonlinear convex* constraints [10], which can be solved by general convex optimization methods, such as, ellipsoid methods or bundle methods. A different approach to solve the optimization problems in eq.(7) and eq.(11) is considered. The constraints in eq.(8-10) are first converted to *linear convex* constraints, which can be expressed as LMIs. Such that readily available optimization tools can be used. [12] presented a *variable change* scheme through which the magnitude constraints are posed as linear convex constraints. The same scheme can be applied to eq.(7) and in eq.(11). Let $h_0 = [h_0(0), h_0(1), \cdots, h_0(N-1)]^T \in \mathbb{R}^N$ be the lowpass analysis FIR filter coefficient vector such that $h_0(n) = 0$ for $n < 0$, $n \geq N$. The autocorrelation sequence

$$r(i) = \sum_{n=-N+1}^{N-1} h_0(n)h_0(n+i), \quad i \in \mathbb{Z}, \tag{15}$$

has the properties that $r(n) = r(-n)$ and $r(n) = 0$ for $n \geq N$. Therefore, it suffices to specify the autocorrelation coefficients for $0 \leq n \leq N-1$. Furthermore, $r(0)$ is given by

$$r(0) = \sum_{n=0}^{N-1} h_0^2(n) = \|h_0\|_2^2 \tag{16}$$

Define the autocorrelation sequence vector $r = [r(0), r(1), \cdots, r(N-1)]^T$. The Fourier transform of the autocorrelation sequence is

$$R(\omega) = \sum_{i\in\mathbb{Z}} r(i)e^{-j\omega i} = r(0) + \sum_{i=1}^{N-1} 2r(i)\cos\omega i = | H_0(e^{j\omega}) |^2, \tag{17}$$

which equals to the squared magnitude of the filter's spectral response. The autocorrelation sequence vector r can be used as the optimization variable. As the filter coefficient vector h_0 can be derived from $r \in \mathbb{R}^N$ via *spectral factorization* [13]. The optimization problem in eq.(7) can be written in terms of r

$$\text{Minimize} \quad \tilde{\delta} \tag{18}$$

$$\text{Subject to} \quad \begin{pmatrix} \alpha - [R(\omega) + R(\pi-\omega)] & 0 & 0 \\ 0 & R(\omega) + R(\pi-\omega) & 1 \\ 0 & 1 & \alpha \end{pmatrix} \succeq 0,$$

$$\omega \in [0, \frac{\pi}{2}], \tag{19}$$

$$R(\omega) \leq \tilde{\delta}, \quad \omega \in [\omega_s, \pi], \tag{20}$$

$$R(\omega) \geq 0, \quad \omega \in [0, \pi]. \tag{21}$$

According to the definition of positive definite matrix, the constraint in eq.(19) is equivalent to

$$\left.\begin{array}{l} R(\omega) + R(\pi-\omega) \leq \alpha \\ \alpha[R(\omega) + R(\pi-\omega)] \geq 1 \end{array}\right\} \longrightarrow \frac{1}{\alpha} \leq R(\omega) + R(\pi-\omega) \leq \alpha, \tag{22}$$

which is actually the constraint eq.(8), since $R(\omega) = | H(e^{j\omega}) |^2$. The constraints in eq.(20) and eq.(21) are gotten by directly substituting $| H(e^{j\omega}) |^2$ with $R(\omega)$. Note that $R(\omega)$ can be expressed in terms of the optimization variables as

$$R(\omega) = [0, 1, 2\cos\omega, \cdots, 2\cos\omega(N-1)] \times \begin{bmatrix} \tilde{\delta} \\ r(0) \\ r(1) \\ \vdots \\ r(N-1) \end{bmatrix}. \tag{23}$$

Thus, the constraints in eq.(20-22) are all linear functions of the optimization variable while the linear functions are always convex. As a result, the LMI constraints of the optimization problems in eq.(18) are convex, since sampling preserves convexity. With convexity guaranteed, the semi-definite program for LMI constraints will converge to the *global* optimal solution. The constraint in eq.(21) is a necessary and sufficient condition for the existence of the solution $h_0 \in \mathbb{R}^N$. It is stated in [13] that there exists $h_0 \in \mathbb{R}^N$ whose autocorrelation sequence vector is $r \in \mathbb{R}^N$ if and only if

$$R(\omega) \geq 0, \quad \omega \in [0, \pi]. \tag{24}$$

As a result, not all $r \in \mathbb{R}^N$ is a valid autocorrelation sequence vector of FIR filters. A variation of eq.(11) can be obtained,

$$\text{Minimize} \quad \tilde{\alpha} \tag{25}$$

$$\text{Subject to} \quad \begin{pmatrix} \tilde{\alpha} - [R(\omega) + R(\pi-\omega)] & 0 & 0 \\ 0 & R(\omega) + R(\pi-\omega) & 1 \\ 0 & 1 & \tilde{\alpha} \end{pmatrix} \succeq 0,$$

$$\omega \in [0, \frac{\pi}{2}], \tag{26}$$

$$R(\omega) \leq \delta, \quad \omega \in [\omega_s, \pi], \tag{27}$$

$$R(\omega) \geq 0, \quad \omega \in [0, \pi]. \tag{28}$$

The equivalence of the optimization problems in eq.(11), and that in eq.(25) can be proved in a manner similar to that proofing the equivalence of the optimization problems in eq.(7) and eq.(18). Note that the term $\tilde{\alpha}[R(\omega) + R(\pi-\omega)]$ in the constraint in eq.(26) is not a linear function of the optimization variable $[\tilde{\alpha}, r^T]^T$. However it is still convex with respect to the optimization variable. The LMI constraints in eq.(26-28) are all convex. As a result, globally optimal solution can be obtained. Alternatively, we can choose to minimize the autocorrrelation coefficient $r(0)$. Since $r(0)$ equals to the ℓ_2 norm of the FIR lowpass filter $H_0(z)$. As a result, the designed QMF banks will be ℓ_2 optimal, such that the subband filter has the smallest total spectral energy with a constrained reconstruction error. This is equivalent to design subband filters with least square optimal stopband energy.

$$\text{Minimize} \quad r(0) \tag{29}$$

$$\text{Subject to} \quad \begin{pmatrix} \alpha - [R(\omega) + R(\pi-\omega)] & 0 & 0 \\ 0 & R(\omega) + R(\pi-\omega) & 1 \\ 0 & 1 & \alpha \end{pmatrix} \succeq 0,$$

$$\omega \in [0, \frac{\pi}{2}], \tag{30}$$

$$R(\omega) \leq \delta, \quad \omega \in [\omega_s, \pi], \tag{31}$$

$$R(\omega) \geq 0, \quad \omega \in [0, \pi]. \tag{32}$$

The convexity of the constraints in eq.(30-32) can be proved in a similar way as that in eq.(18).

Table 1. Design results for *Example 1*

minimization variable	specified parameters			
$\sqrt{\delta}$	N	α	ω_p	ω_s
$0.0027(-51.4538dB)$	30	1.001	0.5π	0.6π

Table 2. Design results for *Example 2*

minimization variable	specified parameters			
α	N	$\sqrt{\delta}$	ω_p	ω_s
1.001054	24	$0.01(-40dB)$	0.5π	0.604π

4. SIMULATION RESULTS

Three design examples are presented, which correspond to the three optimization formulations described in Section 3. Semi-definite programming solver *mincx* in the LMI Control Toolbox of Matlab [7] is employed to solve the linear objective optimization problem with LMI constraints described by eq.(18,25,29).

Example 1: The QMF bank is designed according to eq.(18). The lowpass analysis filter length $N = 30$, and the biggest reconstruction error ripple α is chosen to be 0.001(-60dB) with $\alpha = 1.001$. The passband and the stopband edges of the lowpass analysis filter are $\omega_p = 0.5\pi$ and $\omega_s = 0.6\pi$ respectively. The optimization result gives lowpass filter with maximum stopband ripple size $\sqrt{\delta} = 0.0027(-51.4538dB)$ as observed in Fig.2(a). Since the L_∞ norm of the stopband ripple size is minimized, the lowpass filter has equi-ripple stopband. The magnitude response of the overall QMF bank is shown in Fig.2(b), where equi-ripple property is observed. For clarity, the parameters and minimizing variable of the lowpass analysis subband filter are tabulated in Table 1.

Example 2: The QMF bank specification is given in Table 2. The optimization problem is formulated as eq.(25) with maximal stopband ripple size $\sqrt{\delta}$ being $0.01(-40dB)$. The resulting biggest reconstruction error ripple is $-59.5dB$ with $\alpha = 1.001054$. The magnitude response of the lowpass analysis filter is shown in Fig.3(a) while that of the overall QMF bank is shown in Fig.3(b), where the actual biggest reconstruction error ripple has been constrained to an acceptable level and is less than 0.00424dB.

Example 3: The parameters of the lowpass analysis subband filter are shown in Table 3. The design is carried out according to the optimization program in eq.(29). As observed from the magnitude response of the lowpass analysis subband filter and the magnitude response of the overall QMF bank as shown in Fig.4(a) and (b) respectively, the stopband attenuation level of individual subband filters has been constrained to less than -40dB and the biggest allowable ripple size of the reconstruction error has been constrained to less than $-3.5 \times 10^{-4}dB$.

The subband filters should have narrow transition band and large stopband attenuation. These characteristics are important to achieve of good spectral separation between subband signals. The better the spectral separation, the better the filter bank performance is expected. However, the narrow transition band characteristic, the large stopband attenuation and small reconstruction error ripples are conflicting objectives. Actually, to widen the transition region has been a common tradeoff trick to get a smaller reconstruction error ripple size. In the above three design examples, the transition band width is specified as $\omega_s - \omega_p = 0.1$, or, 0.104π, which is much narrower than that presented in literatures, such as [11], while the ripple size of the reconstruction error and the stopband ripple size have been constrained to an acceptable level. The tolerable level of the stopband attenuation depends on applications. Typically, the analysis subband filters must have a stopbband attenuation larger than 40dB to reduce the energy leakage between the subbands in subband coding of speech signals. In the above three design examples, the maximal stopband ripple size $\sqrt{\delta}$ has been specified or designed to be less than -40dB. α is responsible for the ripple size of the reconstruction magnitude error, which has been specified or designed to a tolerable level. The designed filter banks in *Example 1* has stopband equi-ripple property while the one in *Example 2* has the smallest allowable maximal ripple size of the reconstruction error with a specified stopband ripple level. The ℓ_2 optimal filter bank in *Example 3* has subband filters with the smallest stopband energy with a constrained reconstruction error.

Table 3. Design results for *Example 3*

variable	parameters				
$r(0)$	N	$\sqrt{\delta}$	α	ω_p	ω_s
0.499962	30	$0.01(40dB)$	1.0001	0.5π	0.6π

5. CONCLUSIONS

Three QMF banks design methods are presented. Two are constrained L_∞ norm optimal design and one is constrained ℓ_2 norm optimal design. Both methods cast the design problems as a linear objective function minimization problem with Linear Matrix Inequality (LMI) constraints, which can be solved by the readily available semi-definite programming tools. The LMI constraints have been shown to be convex and *globally* optimal solutions are achieved. The constrained L_∞ norm optimal design minimizes the maximal stopband ripple size given an allowable maximal ripple size of the reconstruction error, or minimizes the allowable maximal ripple size of the reconstruction error with a specified maximal stopband ripple size. Constrained ℓ_2 norm optimal design results in QMF banks with subband filters that have least square total energy with a given reconstruction error constraint. Under the reconstruction error constraint and the symmetrical property of the spectral response of QMF banks, the stopband energy of the subband filters are therefore least square optimal. Optimal design solutions can be obtained by the presented methods. Furthermore, the design methods are very flexible in integrating extra constraints. Design examples are presented to demonstrate the effectiveness of the methods.

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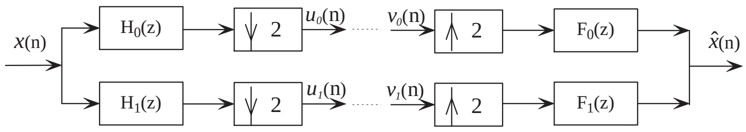

Fig. 1. A two-channel maximally decimated filter banks.

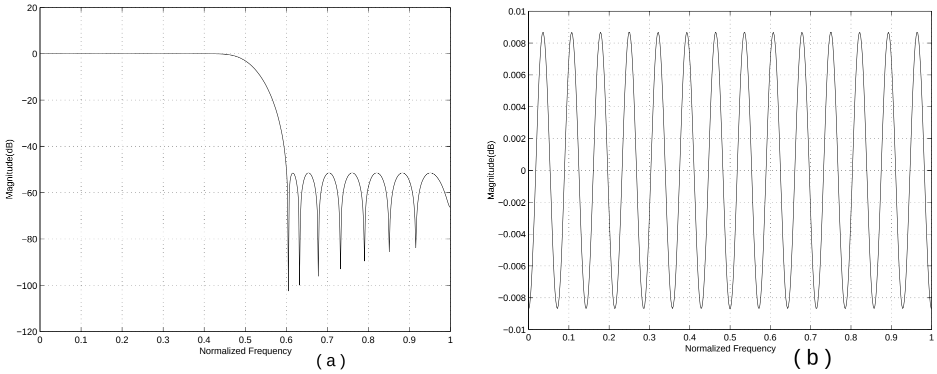

Fig. 2. (a) The magnitude response of stopband equi-ripple lowpass analysis filter; (b) The magnitude response of the overall QMF bank.

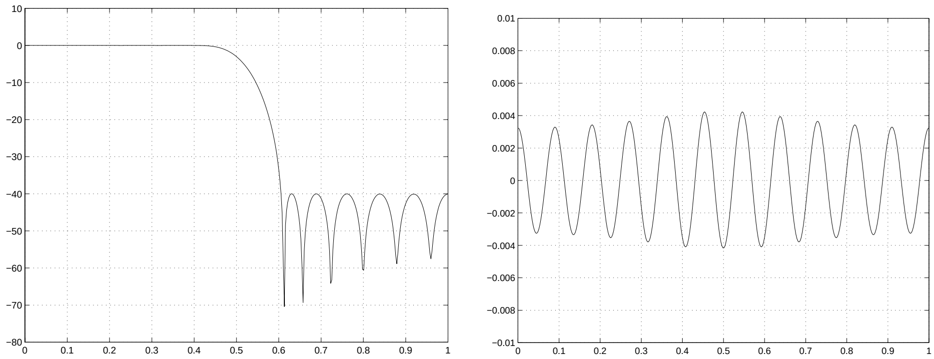

Fig. 3. (a) The magnitude response of lowpass analysis filter; (b) The magnitude response of the overall QMF bank with optimal allowable maximal ripple size.

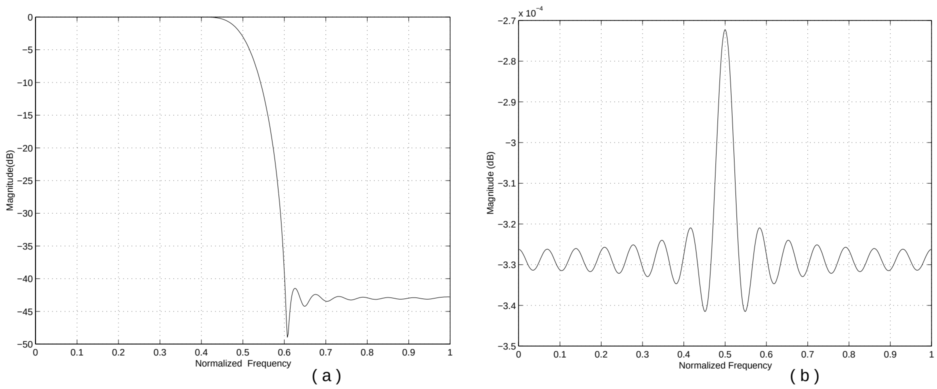

Fig. 4. (a) The magnitude response of ℓ_2 norm optimal lowpass analysis filter; (b) The magnitude response of ℓ_2 optimal QMF bank.