DESIGNING IIR FILTER BANKS COMPOSED OF ALLPASS SECTIONS

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ABSTRACT

In this paper we discuss the design of passband linear phase IIR filter banks composed of allpass sections which are structurally constrained to have low coefficient sensitivity and low roundoff noise level. The design problem is formulated as H_∞ norm minimization which is subsequently converted to a series of LMIs and solved by semidefinite programming. Simulation results shown that subband filters with passband linear phase property and high stopband attenuation are obtained, and the whole filter banks has small reconstruction error.

1. INTRODUCTION

Consider an analysis subband filter pair with transfer functions as shown in the block diagram in Figure 1,

$$H_0(z) = 0.5[A_0(z^2) + z^{-1}A_1(z^2)], \quad (1)$$
$$H_1(z) = 0.5[A_0(z^2) - z^{-1}A_1(z^2)], \quad (2)$$

with $A_0(z)$ and $A_1(z)$ being stable allpass filters of orders K_0 and K_1 respectively. The parallel connection of the two allpass filters described in Figure 1 offers one of the best structures for IIR filter implementation. Such filter structures have low coefficient sensitivity and a low roundoff noise level [1]. Furthermore, they can be realized efficiently by using first and second-order allpass sections as the basic building blocks. The resulting filter structures are highly modular, making them suitable for signal processor and VLSI implementations.

The above filter structure was adopted to design and implement 2-channel filter banks as shown in Figure 2, where E_0, E_1, R_0 and R_1 are allpass filters. The subband filters of such filter banks inherit the advantages of low coefficient sensitivity and low roundoff noise level. However, designing the allpass filters for such filter banks is not easy. The focus of much research activity has been on searching for an algorithm to design such filter banks. The design method in [2] suggested having $A_0(z) = A_1(z^{-1})$, and the analysis and synthesis subband filters quadrature related so that they form an orthogonal filter bank, with a structurally perfect reconstruction property. However, such structural constraints have left a few degree of freedom to design filter banks with desired properties, such as high stopband attenuation etc [3]. The orthogonal structure [2] is relaxed in [4], so that only the quadrature condition is retained. That better filter banks can be obtained is shown in [4]. However, the design method is complicated and this results various design approximations being proposed in the literature. For example, in [5], instead of designing the allpass filter directly, a FIR filter is first designed and this is then converted to an allpass filter. A different approximation approach is considered in [4], where the phase response of the filter is approximated using an orthogonal series. A close form expression of the subband filter coefficients is presented in [4]. In a different approach, [6] considered the case of $A_0(z)$ as a delay function. Only the phase response of $A_1(z)$ needs to be designed. An eigenfilter design method is used to design $A_1(z)$.

In our research, the allpass filter design problem for 2-channel filter banks is converted into an H_∞ optimization problem. This was done so that the readily available semidefinite programming software tool could be used to solve the optimization problem. The proposed formulation was straight forward and resulted in good performance filter banks with high stopbband attenuation, and small spectral overlap subband filters compared to those in [2, 3, 4, 5, 6]. Furthermore, the proposed design method does not impose any relationship between the analysis and synthesis subband filters, nor between the allpass filters that form the subband filters. Design examples are presented when the allpass filters that result in linear phase analysis subband filters are used. Linear phase synthesis subband filters with high stopband attenuations are obtained through the use of the proposed design method. A last note, the subband filters shown in Figure 2 are structurally imposed as mirror images in the frequency domain. As a result, the good spectral property of one subband filter, such as high stopband attenuation, results in filter banks with a good overall spectral response.

2. TWO CHANNEL FILTER BANKS

The structure shown in Figure 2 has the reconstructed signal $\hat{x}(n)$. This relates to the input signal $x(n)$ as

$$\begin{aligned}\hat{X}(z) &= 0.5(H_0(z)F_0(z) + H_1(z)F_1(z))X(z) \\ &\quad +0.5(H_0(-z)F_0(z) + H_1(-z)F_1(z))X(-z) \\ &= T(z)X(z) + A(z)X(-z), \end{aligned} \quad (3)$$

where $T(z) = 0.5(H_0(z)F_0(z)+H_1(z)F_1(z))$ is the linear transfer function of the filter banks. $A(z) = 0.5(H_0(-z)F_0(z) + H_1(-z)F_1(z))$ is the aliasing component of the filter banks. The filter banks are perfect reconstruction when $A(z) = 0$, and $T(z) = cz^{-\ell}$ for some nonzero $c \in \mathbb{R}$ and $\ell \in \mathbb{Z}$, such that the filter banks form a scaled delay system.

Traditionally, the analysis and synthesis subband filters are QM (quadrature mirror) related. This is because the mirror filter bank has a lower implementation complexity compared to systems with unrelated lowpass and highpass filters [8]. Secondly, the aliasing component, $A(z)$, is structurally eliminated when the analysis and synthesis filter banks are quadraturally related. Instead of following the QMF design, we considered the filter banks shown in Figure 2, where the subband filters are pairwise mirror related with $H_1(z) = H_0(-z)$ and $F_1(z) = -F_0(-z)$. The polyphase realization of the analysis and synthesis subband filters are given by

$$H_0(z) = E_0(z^2) + z^{-1}E_1(z^2), \quad H_1(z) = E_0(z^2) - z^{-1}E_1(z^2), \quad (4)$$
$$F_0(z) = R_1(z^2) + z^{-1}R_0(z^2), F_1(z) = -R_1(z^2) + z^{-1}R_0(z^2). \quad (5)$$

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The linear term $T(z)$, and aliasing term $A(z)$ are given by

$$T(z) = z^{-1}(E_0(z^2)R_0(z^2) + E_1(z^2)R_1(z^2)), \quad (6)$$

and

$$A(z) = z^{-1}(E_0(z^2)R_0(z^2) - E_1(z^2)R_1(z^2)). \quad (7)$$

It is observed from eq.(6) and eq.(7) that if

$$E_i(z)R_i(z) \approx z^{-K}, \quad i = 0, 1, \quad (8)$$

is satisfied, we will obtain

$$T(z) \approx z^{-(2K+1)}, \quad (9)$$

$$A(z) \approx 0. \quad (10)$$

As a result, a perfect reconstruction system is realized.

3. H_∞ MODEL-MATCHING DESIGN

If $E_i(z)$ and $R_i(z)$, $i = 0, 1$ are imposed as the allpass functions. The filter banks described by eq.(4) and eq.(5) will inherit all the nice implementation properties described in the Introduction. Assume that the allpass filters $E_i(z)$ and $R_i(z)$, $i = 0, 1$, have a representation as

$$E_i(z) = \frac{N_{E_i}(z)}{D_{E_i}(z)} = \frac{N_{E_i}(z)}{z^{-L_{E_i}} N_{E_i}(z^{-1})}, \quad i = 0, 1, \quad (11)$$

$$R_i(z) = \frac{N_{R_i}(z)}{D_{R_i}(z)} = \frac{\frac{A_i(z)}{B_i(z)}}{\frac{z^{-L_{A_i}} A_i(z^{-1})}{z^{-L_{B_i}} B_i(z^{-1})}}, \quad i = 0, 1. \quad (12)$$

where L_{E_i}, L_{A_i} and L_{B_i} are the orders of polynomials $N_{E_i}(z)$, $A_i(z)$ and $B_i(z)$ respectively. From eq.(12), if $L_{A_i} = L_{B_i}$, then the polyphase components $R_i(z)$, $i = 0, 1$, can be written as

$$R_i(z) = \frac{A_i(z)B_i(z^{-1})}{B_i(z)A_i(z^{-1})}, \quad i = 0, 1, \quad (13)$$

which are allpass functions. As a result, with specified polyphase components $N_{E_i}(z)$, $i = 0, 1$, the polyphase components $R_i(z)$, $i = 0, 1$ can be achieved by

$$N_{E_i}(z)\frac{A_i(z)}{B_i(z)} \approx z^{-m}, \quad i = 0, 1, \quad (14)$$

with $m = \frac{K - L_{E_i}}{2}$, $i = 0, 1$. Note that eq.(8) is satisfied when eq.(14) is fullfilled. As a result, the perfect reconstruction property of the filter banks system is realized. Eq.(14) can be viewed as a traditional equalization problem between $N_{E_i}(z)$ and $N_{R_i}(z) = \frac{A_i(z)}{B_i(z)}$. The equalization problem can be formulated as a H_∞ model matching problem,

$$\min \gamma_i \quad : \quad \gamma_i = \|0.5z^{-m} - N_{E_i}(z)N_{R_i}(z)\|_\infty \qquad \forall\, i = 0, 1. \quad (15)$$

The above model-matching problem can be transformed to the standard controller design problem [9] as shown in Figure 3. The transfer function of the plant is given by

$$G_i(z) = \begin{pmatrix} W(z) & -I \\ N_{E_i}(z) & 0 \end{pmatrix}, \qquad \forall\, i = 0, 1, \quad (16)$$

with $W(z) = 0.5z^{-m}$ being the ideal response of the equalized system in eq.(15). The solution of the full-order H_∞ model matching problem can be obtained by solving a Riccati equation. However, the standard H_∞ approaches result in filters of order equal to the order of the system. However, reduced-order filters, i.e., filters of order lower than the order of the system, are often desirable to reduce the complexity and computational burden of the real-time processing. Reduced-order H_∞ model-matching problem have been considered in [15] but the general problem has not been addressed. Reduced order H_∞ model-matching problem results in nonconvex problems that can be solved with linear matrix inequalities (LMI) [17][16]. The LMI can be solved efficiently and exactly by using semidefinite programmings [10]. Let $\left[\begin{array}{c|c} A_M & B_M \\ \hline C_M & D_M \end{array}\right]$ denotes the system matrix of $M(z) = C_M(zI - A_M)^{-1}B_M + D_M$. The state descriptions of the matrix components in the plant $G_i(z)$ are given by

$$W(z) = \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array}\right], \quad (17)$$

$$N_{E_i}(z) = \left[\begin{array}{c|c} A_{N_{E_i}} & B_{N_{E_i}} \\ \hline C_{N_{E_i}} & D_{N_{E_i}} \end{array}\right], \qquad \forall\, i = 0, 1. \quad (18)$$

The system matrix of the plant $G_i(z)$ in Figure 3 is given by

$$G_i(z) = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] = \left[\begin{array}{cc|cc} A_W & 0 & B_W & 0 \\ 0 & A_{N_{E_i}} & B_{N_{E_i}} & 0 \\ \hline C_W & 0 & D_W & -I \\ 0 & C_{N_{E_i}} & D_{N_{E_i}} & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \qquad \forall\, i = 0, 1. \quad (19)$$

The H_∞ norm γ_i in eq.(15) can be minimized over $R = R^T$ and $S = S^T$, while R and S satisfy [11, 12],

$$\begin{pmatrix} N_{12} & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} AR + RA^T & RC_1^T & B_1 \\ C_1R & -\gamma_i I & D_{11} \\ B_1^T & D_{11}^T & -\gamma_i I \end{pmatrix} \begin{pmatrix} N_{12} & 0 \\ 0 & I \end{pmatrix} < 0, \quad (20)$$

$$\begin{pmatrix} N_{21} & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A^TS + SA & SB_1 & C_1^T \\ B_1^TS & -\gamma_i I & D_{11}^T \\ C_1 & D_{11} & -\gamma_i I \end{pmatrix} \begin{pmatrix} N_{21} & 0 \\ 0 & I \end{pmatrix} < 0, \quad (21)$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0, \quad (22)$$

with N_{12} and N_{21} denotes the bases of the null spaces of (B_2^T, D_{12}^T) and (C_2, D_{21}), respectively. LMIs (20), (21) and (22) are convex constraints on R, S and X, and can be solved using convex programming [10]. The state space parameters of the optimal controller $N_{R_i}(z)$ can be derived from the optimization result (R, S, γ_i) [13]. Existing H_∞ optimization tools, such as *dhinflmi* in the Matlab LMI toolbox [11], can be used to solve the above LMI optimization problem. This method gives the IIR solution $N_{R_i}(z) = \frac{A_i(z)}{B_i(z)}$ with order L_{A_i} equal to L_{B_i}, $i = 0, 1$. Given system delay $K_D = 2K + 1$ and order L_{E_i}, the order of the synthesis polyphase components are both equal to $\frac{K + L_{E_i}}{2}$.

4. NEAR PHASE-ERROR-FREE PROPERTY

Although the optimization problem in eq.(14) deals with the polyphase components of the subband filters, it can be shown that the obtained synthesis filter bank has linear phase property with a given linear phase analysis filter bank. As derived in [14], the product of the polyphase components $E_0(z)E_1(z)$ of a linear phase FIR or IIR filter $H(z) = E_0(z^2) + z^{-1}E_1(z^2)$ is either symmetric or antisymmetric. Assume the H_∞ optimization problem in eq.(14) has achieved a reasonably good solution. As a result, eq.(8) is satisfied. Substituting eq.(8) into eq.(5) yields a pair of synthesis subband filters,

$$F_0(z) \approx z^{-2K}\frac{E_0(z^2)+z^{-1}E_1(z^2)}{E_0(z^2)E_1(z^2)} = z^{-2K}\frac{H_0(z)}{E_0(z^2)E_1(z^2)},$$

$$F_1(z) \approx -z^{-2K}\frac{E_0(z^2)-z^{-1}E_1(z^2)}{E_0(z^2)E_1(z^2)} = -z^{-2K}\frac{H_1(z)}{E_0(z^2)E_1(z^2)}.$$

Since $H_0(z)$ and $H_1(z)$ are linear phase filters and the product of the polyphase components $E_0(z)E_1(z)$ is either symmetric or antisymmetric, the synthesis subband filters have linear phase responses. In practice, the H_∞ optimization problem will achieve good approximation in the passband region of the subband filter. As a result, the design examples presented in Section V have good passband linear phase properties. The passband linear phase property effectively supports the linear phase property of the overall filter banks.

5. DESIGN EXAMPLES

Two examples are presented in this section to demonstrate the effectiveness of the proposed design method.

Example 1: The IIR analysis filter bank considered in [3] with linear phase analysis lowpass filter $H_0(z) = z^{-4}(A(z^2)+z^{-1}A(z^{-2}))$, where $A(z) = \frac{1+az^{-1}+bz^{-2}}{b+az^{-1}+z^{-2}}$ with $a = 5.50439940744155$ and $b = 2.07205705612382$, is applied. The magnitude response of the allpass analysis filters is shown in Figure 4(a). The synthesis filter bank is designed in such a way that the overall filter banks system has a system delay equals to $2K+1 = 29$. The magnitude, and phase response of the designed synthesis filter bank are shown in Figures 4(b) and (c), respectively. The stopband attenuation of the designed synthesis subband filter is larger than 45.7dB and the phase response is passband linear across the spectrum. The magnitude response of the overall filter banks system, shown in Figure 4(e), has a ripple size smaller than 8×10^{-5} dB . Furthermore, the synthesis filters have linear phase properties, as can be observed in Figure 4(f). The aliasing error of the overall filter banks system is less than -40dB, as shown in Figure 4(d).

Example 2: The IIR lowpass filter considered in [7] is applied to construct the analysis lowpass subband filter $H_0(z) = \frac{0.105573+z^{-1}}{1+0.105573z^{-1}} + z^{-1}\frac{0.527864+z^{-1}}{1+0.527864z^{-1}}$. The magnitude and phase responses of the analysis filter bank are shown in Figures 5(a) and (b) respectively. The synthesis filter bank is designed to achieve an overall filter banks system delay equals to $2K+1 = 31$. The magnitude response of the designed synthesis filter banks, shown in Figure 5(c), has approximately 40dB stopband attenuation. Moreover, the synthesis subband filters have linear phase responses in the passband, as can be observed in Figure 5(d). The passband linear phase property effectively supports the overall filter banks system to has a linear phase response as shown in Figure 5(g). The magnitude response of the overall filter banks system achieves a maximal ripple size smaller than 3.5×10^{-4} dB, as shown in Figure 5(f). The aliasing error of the overall filter banks system, shown in Figure 5(e), is smaller than -33dB.

6. CONCLUSIONS

A method for designing IIR filter banks with a passband linear phase properties was presented. The designed subband filters were composed with allpass sections which were structurally constrained to have low coefficient sensitivity and low roundoff noise level. The design problem was formulated as H_∞ minimization of a model matching problem. The H_∞ optimization problem was subsequently converted to a series of LMIs and solved using semidefinite programming. Two design examples were presented, where the analysis filter banks were formed by linear phase and passband linear phase filters in the literature. The designed synthesis filter banks have high stopband attenuation, good spectral separation and linear phase responses. Small reconstruction errors and linear phase properties are observed from the overall filter banks' response in both examples. Both examples have shown to achieve better synthesis filter banks compared to those discussed in the literature. The proposed design method can be extended to multichannel filter banks using the method suggested in [7].

7. REFERENCES

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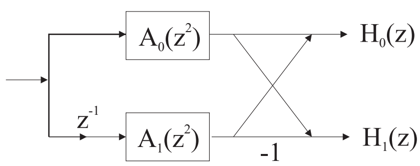

Fig. 1. Lowpass and highpass filter pairs composed using parallel allpass sections.

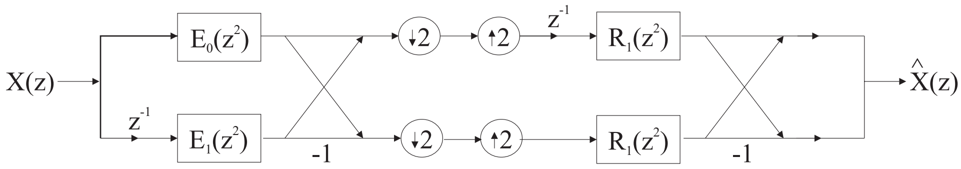

Fig. 2. 2 channel filter banks composed by parallel allpass sections.

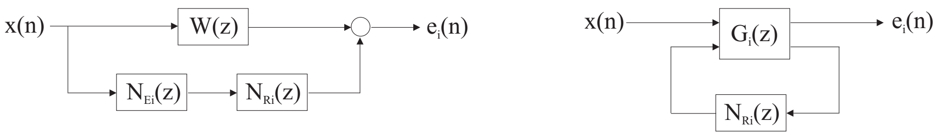

Fig. 3. The synthesis filter polyphase component design formulated as model-matching problem, and its equivalent controller design problem

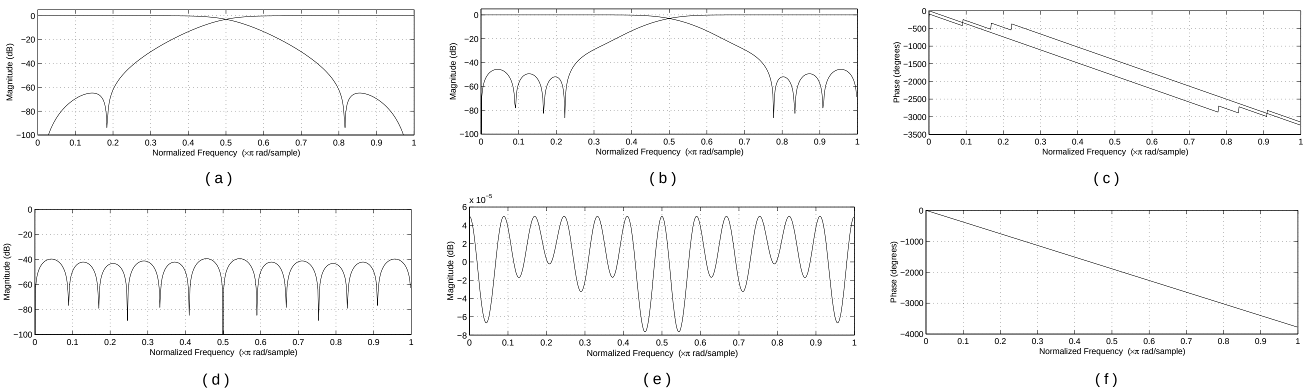

Fig. 4. The magnitude response of (a) the linear phase IIR analysis filter bank and (b) the synthesis filter bank. (c) The phase response of the synthesis filter bank. (d) the aliasing error and (e) The magnitude response, and (f) the phase response of the filter banks.

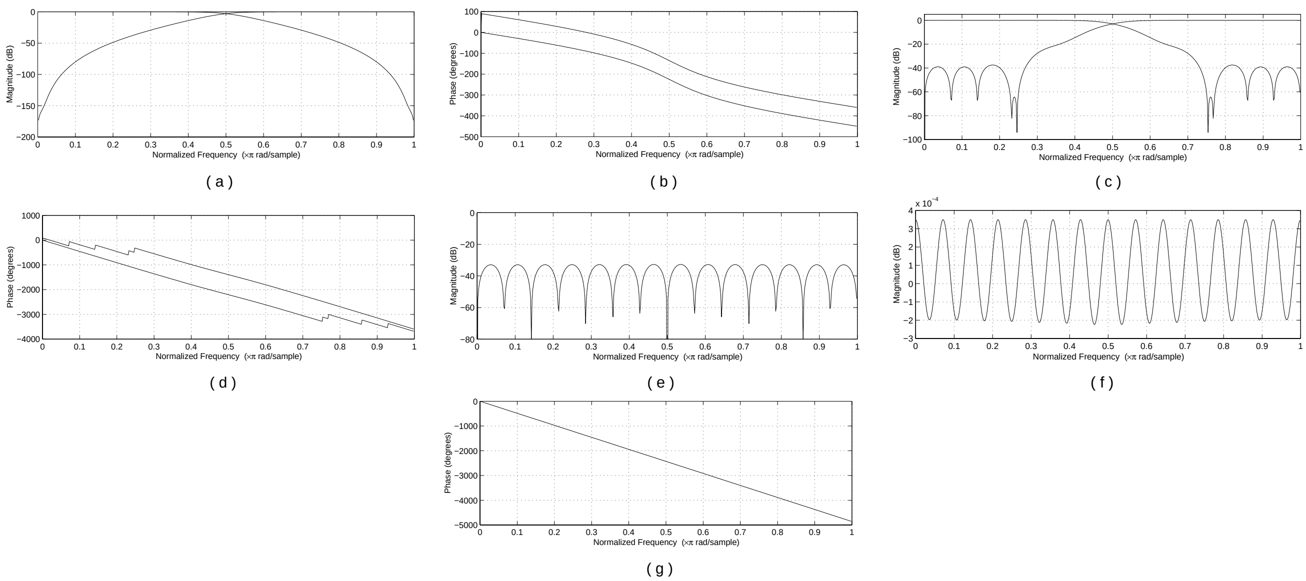

Fig. 5. The magnitude response of (a) the linear phase IIR analysis filter bank and (c) the synthesis filter bank. The phase response of (b) the analysis filter bank and (d) the synthesis filter bank. (e) The aliasing error, (f) the magnitude response and (g) the phase response of the filter banks.