

# A SIMPLE SCHEME FOR ENHANCED REASSIGNMENT OF THE SMOOTHED PSEUDO WIGNER-VILLE REPRESENTATION OF NOISY SIGNALS

*Artur Łoza, Nishan Canagarajah and Dave Bull*

Centre for Communications Research, University of Bristol  
Merchant Ventures Building, Woodland Road, Bristol BS8 1UB, UK

## ABSTRACT

The reassignment procedure has been often employed to improve readability of some Time-Frequency Representations (TFRs). When processing noisy signals, the problem of sensitivity of the technique to noise is encountered. In this paper, a simple modification of reassignment method is proposed, based on thresholding operation. Specifically, by preventing the reassignment of the distribution coefficients below the noise dependent threshold and replacing them with zeros, the enhanced signatures on the time-frequency plane are obtained. This method is compared with other techniques, such as the Reassigned Spectrogram (RSP) and the Supervised Reassigned Spectrogram (SRSP). An experimental test of these algorithms as the Instantaneous Frequency (IF) estimators for a chirp signal have shown that our method improves the accuracy of the estimation for heavy noise.

## 1. INTRODUCTION

In practical Time-Frequency (T-F) analysis, signals corrupted by noise are commonly encountered, with the additive Gaussian White Noise (GWN) as one possible model of distortion. In such cases, the problem of the TFR's sensitivity to the presence of noise arise. Previous studies on TFRs of noisy signals considered the kernels for minimum variance representations [1], [2], the robust versions of the Wigner Ville distribution (WV) [3], [4], and the Short Time Fourier Transform (STFT) [4]. In [4], realization of T-F distributions based on the mean and median is presented, with the first solution shown to be suitable for GWN, and the latter for impulse noise. An example of the analysis of the TFRs as IF estimators of noisy signals is presented in [5].

Few publications address the problem of reassigned representations of noisy signals. Reassignment, first introduced in [6], is a non-linear technique which aims to improve resolution in the T-F or Time-Scale (T-S) domain [7]. Although for signal-only sequences the benefits of this method are clear, this technique appears to be highly sensitive to

noise and yields sharp peaks in regions where conventional smoothing would flatten the distribution. In [8], a robust technique, the SRSP is proposed: an improvement to the original method, introduced by a multi-window extension of the procedure.

The methods mentioned so far usually required some modifications of the kernel or the use of several kernels during the smoothing process. These modifications may limit the use of "signal matched" kernels designed for extraction of particular features of a signal. The use of more than one smoothing window increases the computational load of the procedure. Also, the choice of an appropriate window combination for a signal with unknown characteristics, or containing components other than FM signals (e.g. T-F atoms, transients), is not trivial.

In some applications the reassignment technique is used to modify the TFR, to improve performance of a feature extraction or classification system. In such cases, the removal of the noise and the reduction of the dimensionality of a problem is desirable. As mentioned before, conventional reassignment is sensitive to noise and thus additional refinement technique is required. In this paper, we propose a simple modification of the reassignment procedure by introducing a thresholding operation to the reassignment algorithm itself. This prevents the procedure from reassigning noisy regions of the T-F plane into sharp ridges, while still performing reallocation of the signal component coefficients. This modification can be easily extended to other transforms for which a reassignment procedure exists. The method improves the accuracy of the IF estimation for a noisy signal and has potentially lower computational complexity than other approaches.

## 2. METHOD DESCRIPTION

### 2.1. The SPWV and its reassigned version

All Cohen Class distributions can be written as the double convolution of the WV of the signal and a two dimensional

---

This work is supported by Alenia-Marconi Systems UK.

T-F smoothing function  $L$  [9]:

$$TFR_x(t, \omega; L) = WV_x(t, \omega) * L(t, \omega) = \iint WV_x(t', \omega') L(t - t', \omega - \omega') dt' \frac{d\omega'}{2\pi}, \quad (1)$$

where  $WV_x$ , is defined as

$$WV_x(t, \omega) = \int x\left(t + \frac{t'}{2}\right) x^*\left(t - \frac{t'}{2}\right) e^{-i\omega t'} dt'.$$

Different TFRs can be obtained from the fundamental WV distribution by applying a different smoothing function  $L$ . In our experiments a Gaussian separable 2-D function has been used, with the time and frequency widths as parameters. A Gaussian function is used for its optimum T-F localisation property [9]. Here, instead of defining each width separately, we first choose one of the lengths,  $\alpha_0$ , and adjust the volume of the kernel with parameter  $v$ , in the spirit of coupled smoothing:

$$L(t, \omega) = g(t)H(\omega) = \frac{1}{v^2} e^{-t^2/\alpha^2 - \omega^2/\beta^2}, \quad (2)$$

where

$$\alpha = v\alpha_0, \quad \beta = \frac{v}{\alpha_0}. \quad (3)$$

The positive distributions are obtained for  $v \geq 1$  [9]. Throughout this paper  $v = 1$  is used, so that the smoothed distribution is equivalent to the Spectrogram (SP). In such cases,  $L$  is the WV distribution of the SP smoothing window of width  $\sqrt{2}\alpha_0$ .

Reassignment consists of shifting coefficients of a representation in the T-F plane using an appropriate prescription for the displacements [7]:

$$RTFR_x(t', \omega'; L) = \iint TFR_x(t, \omega) \delta(t' - \hat{t}_x(t, \omega)) \delta(\omega' - \hat{\omega}_x(t, \omega)) dt \frac{d\omega}{2\pi}. \quad (4)$$

Coordinates of the reassignment for the Smooth Pseudo Wigner-Ville distribution (SPWV) are computed with two additional TFRs. Since the WV distribution is smoothed directly in (1), we can ‘reuse’ the WV and together with two additional smoothing functions,

$$\begin{aligned} L^{\hat{t}}(t, \omega) &= tL(t, \omega) \\ L^{\hat{\omega}}(t, \omega) &= \omega L(t, \omega), \end{aligned}$$

substitute them back into (1):

$$TFR_x(t, \omega; L^{\hat{t}}) = WV_x(t, \omega) * L^{\hat{t}}(t, \omega) \quad (5)$$

$$TFR_x(t, \omega; L^{\hat{\omega}}) = WV_x(t, \omega) * L^{\hat{\omega}}(t, \omega). \quad (6)$$

The ratio of resulting distributions is then used to compute the reallocations [7]:

$$\hat{t}(t, \omega) = t - \frac{TFR_x(t, \omega; L^{\hat{t}})}{TFR_x(t, \omega; L)} \quad (7)$$

$$\hat{\omega}(t, \omega) = \omega + j \frac{TFR_x(t, \omega; L^{\hat{\omega}})}{TFR_x(t, \omega; L)}. \quad (8)$$

## 2.2. Noise rejection procedure

We consider the signal model

$$x(t_k) = f(t_k) + w(t_k),$$

where both components, a deterministic signal  $f(t_k)$ , and the noise  $w(t_k)$ , are analytic. The analytic noise can be written as  $w(t_k) = w_r(t_k) + jw_H(t_k)$ , where  $w_r(t_k)$  is a real GWN noise with variance  $\sigma_w^2/2$  and  $w_H(t_k)$  is the Hilbert transform of  $w_r(t_k)$  [2]. Throughout this paper,  $t_k$  and  $\omega_l$  will stand for appropriately discretised time and frequency coordinates.

The noise rejection procedure consists of preventing the noisy parts of the distribution from being reassigned and possibly modifying them. Following the discrete algorithm derived in [7], we replace negligible energy thresholding with a decision step. Specifically, having computed T-F distribution  $TFR_x(t_k, \omega_l)$ , we construct the rejection area  $\mathcal{B}$ , basing on a threshold  $\epsilon(t_k, \omega_l)$ :

$$\mathcal{B} = \{(t_k, \omega_l) : |TFR_x(t_k, \omega_l)| < \epsilon(t_k, \omega_l)\},$$

and then, at every point on the T-F plane  $(t_k, \omega_l)$ , we perform one of the two operations on a coefficient  $TFR(t_k, \omega_l)$  depending on the pre-defined constraints:

$$\begin{aligned} RTFR(t_k, \omega_l) &= RTFR(t_k, \omega_l) \\ &\quad + a(t_k, \omega_l) TFR(t_k, \omega_l) \\ &\quad \text{if } (t_k, \omega_l) \in \mathcal{B}, \\ RTFR(\hat{t}_k, \hat{\omega}_l) &= RTFR(\hat{t}_k, \hat{\omega}_l) \\ &\quad + a(t_k, \omega_l) TFR(t_k, \omega_l) \\ &\quad \text{otherwise.} \end{aligned}$$

The second operation is a reassignment step. The first operation depends on pre-defined values of  $RTFR(t_k, \omega_k)$  and  $a(t_k, \omega_l)$ . Here, we initialize the  $RTFR(t_k, \omega_k)$  to zeros as in [7], and replace the rejected noise coefficients with zeros, i.e.  $a(t_k, \omega_l) = 0$  for  $(t_k, \omega_l) \in \mathcal{B}$ , and  $a(t_k, \omega_l) = 1$  otherwise. As a result, only coefficients  $TFR(t_k, \omega_k)$  above the threshold  $\epsilon(t_k, \omega_l)$  will be reassigned, as opposed to the conventional method that reassigned all the coefficients. The area  $\mathcal{B}$  may be thought of as a decision map analogous to that of [8], but computed using different criteria and derived from only one realisation of the distribution. Depending on the nature of signal, different estimates of a threshold  $\epsilon$

are possible. Here, we assume unknown noise level, and a deterministic nature for the signal. For the purpose of enhancement of the output reassignment representation, a global threshold value  $\epsilon(t_k, \omega_l) = \epsilon = \text{mean}\{TFR(t_k, \omega_l)\}$  was chosen. Since the representations we have considered are energy distributions and the signals have zero mean, we can expect  $\epsilon = \text{var}\{x(t)\} = \text{mean}\{|s(t)|^2\} + \sigma_w^2$ . It should be noted that this approach does not preserve the energy of the noisy signal since some of the samples are set to zero. The potential of a two-dimensional local threshold mask and different values of  $a(t_k, \omega_k)$  will be a subject of further investigation.

It is clear that, apart from noise removal, this technique is expected to reduce the number of required operations and computing time, depending on the signal, the extent of the noise regions and the threshold. The number of operations is reduced by the number of coefficients below the threshold. If the threshold is set to zero the complexity of the algorithm is comparable to that of the conventional technique. Moreover, depending on the implementation, the use of the same WV matrix to compute reassignment coordinates as in (5)–(6), may result in a further reduction of computational load.

### 3. PERFORMANCE TESTS

#### 3.1. IF estimation

For analysis of the developed method as an IF estimator, we have chosen a chirp signal corrupted by additive GWN noise as described in Section 2.2:

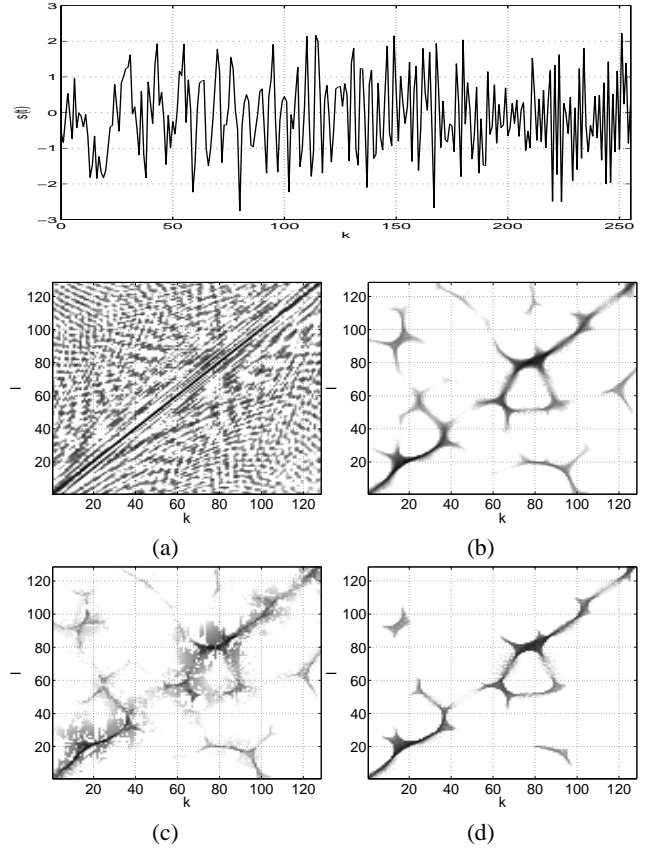
$$x(t) = Ae^{j(ct^2)} + w(t).$$

In this simulation  $t = [0, 1]$ , with sampling period  $T = 1/N$ . The number of samples used was  $N = 256$ . The input SNR is defined as  $\text{SNR}_{in} = 10 \log_{10} A^2/\sigma^2$ . To avoid discretisation error, the value of  $c = N\pi/2$  was chosen so that the IF,  $\omega(t) = 2ct$ , lies on points of the discrete T-F grid.

The performance was assessed in terms of the mean squared error of the IF estimation,  $E\{(\omega(t) - \hat{\omega}(t))^2\}$ , with the IF estimator

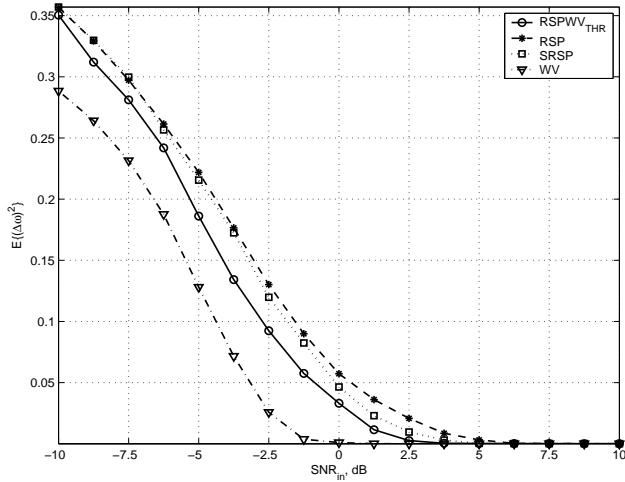
$$\hat{\omega}(t) = \arg\{\max_{\omega} \{TFR(t, \omega)\}\}.$$

The TFRs used in the comparison were: the WV, the RSP, the SRSP [8], and the developed method implemented as the SPWV with the kernel defined in (2)–(3) (see Fig. 1 for examples of the distributions). Smoothing windows were chosen to give equivalent non-reassigned distributions. The noise sequences with  $\text{SNR}_{in} = [-10 \dots 10]$  dB with 1.25 dB step were added to the test signal and 50 simulations were performed using the TFRs from the set above. The



**Fig. 1.** Examples of TFRs of noisy signal (top figure, only real part shown) computed used in the simulation ( $\text{SNR}_{in} = 0$  dB): (a) WV, (b) RSP, (c) SRSP, (d) developed method;  $k$  and  $l$  are time and frequency coordinates, respectively. For the sake of visualisation, the dynamic range of the images was limited to 20 dB from peak value and the central part of the image ( $\frac{N}{2} \times \frac{N}{2}$ ) was displayed.

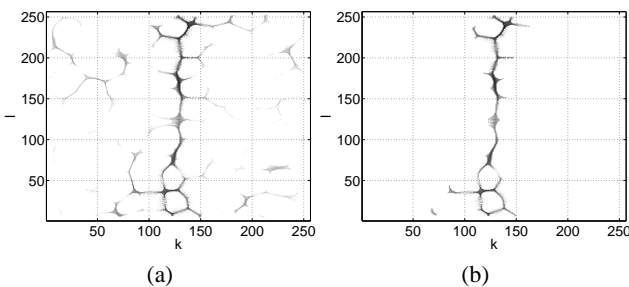
estimation error is shown in Fig. 2. It can be observed that the thresholded reassignment outperforms the conventional RSP and supervised reassignment (SRSP) for the signal with  $\text{SNR}_{in} = -10 \dots 6.25$  dB. For signals with small amount of noise (i.e.  $\text{SNR}_{in} > 6.25$  dB) the thresholding will cause distortion, which contributes to the error. It should be noted, that the WV is the best estimator, as expected for the isolated linear chirp case. It is however known that for other multicomponent signals like, for example, two parallel chirps of finite length, located sufficiently close to each other, the level of cross-components would rule out the WV as the IF estimator using peak value. For example, the estimation error for the signal (mean squared error for the first of the two components):  $x(t) = e^{j(c(t)^2)} + e^{j(c(t+0.125)^2)} + w(t)$  and  $\text{SNR}_{in} = 0$  dB is 0.069 for the WV, whereas for the modified reassignment the error is 0.058.



**Fig. 2.** MSE of IF estimation for a chirp signal corrupted by analytic noise. The developed method has been labelled as  $RSPWV_{THR}$

### 3.2. Noisy radar signal example

Benefits of the thresholding procedure can be more obvious in the case of signals with unknown characteristic. Our aim here was to pre-process automatically representations for subsequent feature extraction. Fig. 3 compares the conventional RSP and modified reassignment representation of a simulated radar return from an unknown object. The SNR for the signal is 7 dB, Peak-SNR is 13.6 dB. The RSP distribution was then thresholded to achieve similar shape of the main lobe of the signal (expected in this radar system in the interval  $k = [97, 160]$ ) and the modified representation was left intact. The developed procedure outputs clear image, whereas conventional reassignment contains reassigned noise above the threshold and thus it would require further processing.



**Fig. 3.** The reassignment comparison for a noisy signal with unknown characteristic: (a) the conventional method, (b) the developed technique.

## 4. CONCLUSIONS

We have proposed a modification of the reassignment method for noisy signals which leads to the enhancement of the output distribution. It has been shown that introducing a thresholding procedure, with the threshold based on a variance of the noisy signal is sufficient to improve the IF estimation for the chirp signal over the conventional method. Further experiments confirmed usefulness of the method for the automated enhancement of T-F images of noisy radar returns. The method is expected to have lower computational complexity than other known reassignment techniques due to re-using the WV matrix and reduction of the number of reallocations in the reassignment procedure.

## 5. REFERENCES

- [1] Moeness G. Amin, "Minimum variance time-frequency distribution kernels for signals in additive noise," *IEEE Transactions on Signal Processing*, vol. 44, no. 9, pp. 2352–2356, September 1996.
- [2] LJubiša Stanković and Veselin Ivanović, "Further results on the minimum variance time-frequency distribution kernels," *IEEE Transactions on Signal Processing*, vol. 45, no. 6, pp. 1650–1655, June 1997.
- [3] LJubiša Stanković, "Algorithm for the Wigner distribution of noisy signals realisation," *Electronics letters, IEE*, vol. 34, no. 7, pp. 622–623, 1998.
- [4] Igor Djurović, Vladimir Katkovnik, and LJubiša Stanković, "Median filter based realizations of the robust time-frequency distributions," *Signal Processing*, vol. 81, pp. 1771–1776, 2001.
- [5] Igor Djurović and LJubiša Stanković, "Influence of high noise on the instantaneous frequency estimation using quadratic time-frequency distributions," *IEEE Signal Processing Letters*, vol. 7, no. 11, pp. 317–319, 2000.
- [6] Kunihiro Kodera, Roger Gendrin, and Claude de Villedary, "Analysis of time-varying signals with small  $BT$  values," *IEEE Transactions on Acoustic, Speech, and Signal Processing*, vol. 26, no. 1, pp. 64–76, February 1978.
- [7] François Auger and Patrick Flandrin, "Improving the readability of time-frequency and time-scale representations by the reassignment method," *IEEE Transactions on Signal Processing*, vol. 43, no. 5, pp. 1068–1089, 1995.
- [8] Eric Chassande-Mottin, *Méthodes de réallocation dans le plan temps-fréquence pour l'analyse et le traitement de signaux non stationnaires*, Thèse de Doctorat, Université de Cergy-Pontoise, <http://www.obs-nice.fr/ecm/ecm/these/ecm98.html>, 1998.
- [9] Leon Cohen, "Time-frequency distributions – a review," *Proceedings of the IEEE*, vol. 77, no. 7, pp. 941–981, July 1989.