THE MINIMAX DESIGN OF DIGITAL ALL-PASS FILTERS WITH PRESCRIBED POLE RADIUS CONSTRAINT USING SEMIDEFINITE PROGRAMMING (SDP)

Carson K. S. Pun, and S. C. Chan

Department of Electrical and Electronic Engineering,
The University of Hong Kong, Pokfulam Road, Hong Kong.

ABSTRACT

This paper proposes a new method for designing digital all-pass filters with a minimax design criterion using semidefinite programming (SDP). The frequency specification is first formulated as a set of linear matrix inequalities (LMI), which is a bilinear function of the filter coefficients and the ripple to be minimized. Unlike other all-pass filter design methods, additional linear constraints can be readily incorporated. The overall design problem turns out to be a quasi-convex constrained optimization problem (solved using the SDP) and it can be solved through a series of convex optimization sub-problems and the bisection search algorithm. The convergence of the algorithm is guaranteed. Nonlinear constraints such as the pole radius constraint of the filters can also formulated as LMIs using the Rouche's theorem. It was found that the pole radius constraint allows an additional tradeoff between the approximation error and the stability margin in finite wordlength implementation. The effectiveness of the proposed method is demonstrated by several design examples.

I. INTRODUCTION

Digital all-pass filters are useful in many applications such as digital communications, phase equalization, implementation of digital and multirate filters, etc [4]. They have a unit magnitude response but a very flexible phase characteristic. Therefore, they are very useful to equalize the phase response of nonlinear phase systems such as IIR filters designed by frequency transformation. They can also be used in efficient realization of digital filters by combining the outputs of two properly design all-pass filters [4]. Due to the symmetry of the numerator and denominator of an all-pass filter, it only requires N multiplications per sample as compared to $2N$ for general IIR filters having the same order of numerator and denominator. From an implementation point of view, digital all-pass filters also possess many nice properties: such as nonlinear oscillation free characteristics, lossless property, and low coefficient sensitivity, which are very important to the finite wordlength implementation with low roundoff noise. These attractive properties have motivated considerable research into the design of digital all-pass filters [1-3,5-7,12]. Weighted least squares designs of all-pass filters were considered in [2,5,6]. Minimax designs of all-pass filters were also considered in [1,3,7,12]. Although it is possible to enforce the stability of the all-pass filters by choosing properly the design specification, the poles of the filters can lie arbitrarily close to the unit circle. Due to the finite wordlength effects in practical implementation, the poles of the transfer function should not lie too close to the unit circle and certain stability margin is desirable [13]. Poles close to the unit circle may also enhance quantization noise. Recently, methods for constraining the maximum pole radius when designing IIR filters have received considerable interest [11,13].

In this paper, we propose a new method for designing causal-stable digital all-pass filters with a minimax design criterion using semidefinite programming. The frequency specification is first formulated as a set of linear matrix inequalities (LMI), which is a bilinear function of the filter coefficients and the ripple to be minimized. Unlike other all-pass filter design methods, additional linear constraints can be readily incorporated. The overall design problem turns out to be a quasi-convex constrained optimization problem (solved using the SDP) and it can be solved through a series of convex optimization sub-problems and the bisection search algorithm. The convergence of the algorithm is guaranteed. Nonlinear constraints such as the pole radius constraint and hence the stability of the filters is formulated as LMIs using the Rouche's theorem. Our formulation is a modification of the method in [13]. We first solve the SDP problem without the pole constraint. If the given pole radius constraint is satisfied, then the solution obtained is the desired result. If not, the radii of those poles exceeding the prescribed pole radius will be reduced to obtain a feasible initial solution. The Rouche's theorem, which will be formulated as a set of LMIs, will be invoked together with the original LMIs to obtain the solution. It was found that the pole radius constraint allows an additional tradeoff between the approximation error and the stability margin in finite wordlength implementation. The effectiveness of the proposed method is demonstrated by several design examples.

The paper is organized as follows: In Section II, a brief introduction to digital all-pass filters is given. Section III is devoted to the proposed design method. The frequency specification is first formulated as a set of matrix inequalities, and is solved successively using the SDP and the bisection search method. Then, the LMI for incorporating the maximum pole radius constraint is derived using the Rouche's theorem. A simple design example is given in section IV and is compared with other design technique. Finally, conclusions are drawn in section V.

II. DIGITAL ALL-PASS FILTERS

The transfer function of a digital all-pass filter, $H_N(z)$, of order N and real-valued filter coefficients can be written as:

$$H_N(z) = z^{-N}\frac{A(z^{-1})}{A(z)},\ a_0 = 1, \tag{2-1}$$

where $A(z) = \sum_{n=0}^{N} a_n z^{-n}$, and a_n 's are real-valued filter coefficients. Substituting $z = e^{j\omega}$ into (2-1), one gets

$$H_N(e^{j\omega}) = e^{-jN\omega} \cdot e^{2j\phi_A(\omega)} = e^{j\theta(\omega)}, \tag{2-2}$$

where $\theta(\omega) = -N\omega + 2\phi_A(\omega)$, and $tan(\phi_A(\omega)) = \frac{\sum_{n=0}^{N} a_n \cdot sin(n\omega)}{\sum_{n=0}^{N} a_n \cdot cos(n\omega)}$. It can be seen that the all-pass filter has a unit magnitude response and its phase response is governed by its filter coefficients. Therefore, it can be used to approximate a given phase response in order to equalize the nonlinear phase response of IIR filters and other systems. They can also be used to implement very efficiently lowpass filters by summing the outputs of two properly designed all-pass filters. Moreover, as the numerator and denominator coefficients are mirror images of each other, it only requires N multiplication for each output sample as opposite to $2N$ for a general IIR filter with the same order of numerator and denominator. Suppose that an all-pass filter is used to approximate a certain desired phase response $\theta(\omega)=\theta_d(\omega)$. To formulate the design problem, we need to express the phase error $\theta_e(\omega)=\theta(\omega)-\theta_d(\omega)$ in terms of the filter coefficients as

$$e^{j\theta_e(\omega)}=\frac{\sum_{n=0}^{N}a_n e^{j\Phi_n(\omega)}}{\sum_{n=0}^{N}a_n e^{-j\Phi_n(\omega)}}, \tag{2-3}$$

where $\Phi_n(\omega)=n\omega-(N\omega+\theta_d(\omega))/2$. Therefore, we have

$$tan(\theta_e(\omega)/2)=\frac{\sum_{n=0}^{N}a_n\cdot sin(\Phi_n(\omega))}{\sum_{n=0}^{N}a_n\cdot cos(\Phi_n(\omega))}. \tag{2-4}$$

(2-4) can be written more compactly in the following matrix form

$$tan(\theta_e(\omega)/2)=\frac{\boldsymbol{a}^T\cdot\boldsymbol{s}(\omega)}{\boldsymbol{a}^T\cdot\boldsymbol{c}(\omega)}, \tag{2-5}$$

where $\boldsymbol{a}=[a_0 \quad a_1 \quad \cdots \quad a_N]^T$,

$\boldsymbol{c}(\omega)=[cos(\Phi_0(\omega)) \quad cos(\Phi_1(\omega))\cdots cos(\Phi_N(\omega))]^T$, and

$\boldsymbol{s}(\omega)=[sin(\Phi_0(\omega)) \quad sin(\Phi_1(\omega))\cdots sin(\Phi_N(\omega))]^T$.

(2-4) characterizes the phase response error of the all-pass filter at frequency ω in terms of the filter coefficients and the desired phase response function. In next section, we will formulate the design problem of the all-pass filters using the minimax error criterion as a quasi-convex optimization problem, which can be solved by a series of convex SDP problems.

III. ALL-PASS FILTER DESIGN USING SDP

3.1 Minimax Error Criterion

If the all-pass filter is used to approximate a desired phase response over a set of disjoint intervals $\omega\in\Omega\subset[-\pi,\pi]$ in a weighted minimax error criterion, then the phase response error is to be bounded above and below by a positive quantity Δ. Since the tangent function is monotonic in the interval $[0,\pi/2]$, the value of $tan(\theta_e(\omega)/2)$ is bounded above and below by $[-\delta,\delta]$, where $\tan(\Delta/2)=\delta$. The design problem at hand is then given by

$$\min_{\boldsymbol{a}}\delta \quad \text{subject to}$$
$$-\delta\le W(\omega)\cdot\frac{\boldsymbol{a}^T\cdot\boldsymbol{s}(\omega)}{\boldsymbol{a}^T\cdot\boldsymbol{c}(\omega)}\le\delta, \text{ for } \omega\in\Omega\subset[-\pi,\pi], \tag{3-1}$$

where $W(\omega)$ is a positive weighting function associated with the disjoint intervals over which the desired response is to be approximated. Note, the denominator in (3-2) cannot be zero and it must assume the same sign over $\omega\in\Omega$. We shall solve (3-2) as a series of SDP feasibility problems. To this end, rewrite (3-1) as:

$$\min_{\boldsymbol{a}}\delta \quad \text{subject to}$$
$$-\delta(\boldsymbol{a}^T\cdot\boldsymbol{c}(\omega))\le W(\omega)\cdot\boldsymbol{a}^T\cdot\boldsymbol{s}(\omega)\le\delta(\boldsymbol{a}^T\cdot\boldsymbol{c}(\omega)). \tag{3-2}$$

We have assumed from stability consideration that the sign of the denominator in (3-1) is positive. (3-2) is not a LMI because the constraints are bilinear function in δ and $\boldsymbol{a}$. To formulate (3-2) as LMIs, we assume that the ripple δ is fix and given. Instead of solving (3-2), we solve the following feasibility problem after discretizing the frequency variable ω in the band of interest Ω into K points, ω_k for $k=1,\cdots,K$,

$$\min_{\boldsymbol{a}}\varepsilon \quad \text{subject to}$$
$$\boldsymbol{F}_k(\boldsymbol{a})+\varepsilon\cdot\boldsymbol{I}_2\ge 0, \text{ for } k=1,\cdots,K, \tag{3-3}$$

where $\boldsymbol{F}_k(\boldsymbol{a})=\begin{bmatrix}\boldsymbol{a}^T\cdot(\delta\cdot\boldsymbol{c}(\omega_k)+W(\omega)\cdot\boldsymbol{s}(\omega_k)) & 0\\ 0 & \boldsymbol{a}^T\cdot(\delta\cdot\boldsymbol{c}(\omega_k)-W(\omega)\cdot\boldsymbol{s}(\omega_k))\end{bmatrix}$,

$\boldsymbol{I}_N$ is an $N\times N$ identity matrix and ε is a fictitious optimization variable introduced to determine whether the given ripple δ can be achieved by the all-pass filter to be designed. If $\varepsilon\le 0$, then the given ripple can be achieved by the all-pass filter. Otherwise, the ripple is too small and it needs to be increased. The constraints, $\boldsymbol{F}_k(\boldsymbol{a})+\varepsilon\cdot\boldsymbol{I}_2\ge 0$, can be stacked together to form the following standard LMI formulation

$$\min_{\boldsymbol{a}}\varepsilon \quad \text{subject to } \boldsymbol{F}(\boldsymbol{a},\varepsilon)\ge 0, \tag{3-4}$$

where $\boldsymbol{F}(\boldsymbol{a},\varepsilon)=diag(\boldsymbol{F}_1(\boldsymbol{a})+\varepsilon\cdot\boldsymbol{I}_2,\boldsymbol{F}_2(\boldsymbol{a})+\varepsilon\cdot\boldsymbol{I}_2,\cdots\boldsymbol{F}_K(\boldsymbol{a})+\varepsilon\cdot\boldsymbol{I}_2)$. Since the matrix $\boldsymbol{F}(\boldsymbol{a},\varepsilon)$ is affine in $\boldsymbol{a}$ and ε, (3-4) can be solved efficiently using SDP. To determine the minimum ripple that can be achieved by the all-pass filter in approximating the given specification, we combine the SDP formulation in (3-4) with the bisection method. More precisely, let δ_L and δ_U be the lower bound and upper bound of δ, respectively (i.e. $\delta_L<\delta<\delta_U$). We first set the ripple to $\delta=(\delta_L+\delta_U)/2$ and solve (3-4) for ε. If the solution of ε is greater than zero, then the given ripple δ is too small and it is set to the new lower bound, i.e. $\delta_L=\delta$. On the other hand, if $\varepsilon\le 0$, then the given ripple is feasible and the upper bound is reduced to $\delta_U=\delta$. After completing the first stage, we repeat this process until $|\delta_L-\delta_U|$ is smaller than a given tolerance. Therefore, the minimum achievable ripple can be found together with the appropriate filter coefficients. In effect, we are solving the design problem as a series of convex SDP problem. Since the difference between the two bounds of the ripples is reduced by half at each iteration, the solution will be found eventually when this difference goes to zero. Thus, the convergence of the algorithm is also guaranteed. Note, additional linear constraints such as a prescribed flatness at a certain frequency can be incorporated.

3.2 Imposing Stability Constraint

The design formulation in section 3.1 only deals with the frequency specification and, like many other proposed methods, it does not enforce the stability constraint. In other words, the all-pass filter so obtained cannot always be stable. The necessary and sufficient conditions for the all-pass filters to be stable are that its group delay satisfies [6,12]

$$\int_0^{\pi}\tau(\omega)d\omega=N\omega \quad \text{where } \tau(\omega)=-\frac{d\theta(\omega)}{d\omega}.$$

An all-pass filter is also stable if it has a phase approximation error less than π at $\omega=\pi$ [6,12]. Several approaches for enforcing the stability constraints in designing IIR filters were previously proposed [13,10,11]. One possibility is to optimize the locations of the poles and require their magnitudes to be smaller than one [10]. In other words, all the poles of the IIR filter are within the unit circle. However, the frequency response of an IIR filter is usually a highly nonlinear function of the pole locations. The constrained optimization can easily be trapped at local minimum. Another method is based on the Lyapunov stability constraint [11], which says that if a digital system is stable, then there exists a positive definite matrix $\boldsymbol{P}$ which satisfies

$$\boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}^T\boldsymbol{A}\ge 0, \tag{3-5}$$

where $\boldsymbol{A}$ is the state transition matrix, and "≥ 0" means that the matrix on the left hand side is positive semidefinite. Notice that (3-5) is bilinear in the parameters $(\boldsymbol{P},\boldsymbol{A})$, which cannot be cast directly as an SDP. The third method, which was proposed in [13], is based on the Rouche's theorem [8,9]. The IIR filter design method in [13] is based on an iterative Gauss-Newton method, where the frequency response of the filter is linearized to obtain a quadratic objective function in the design parameters. The least squares error is then minimized. In using the Rouche's theorem, it is assumed that the filter obtained in the last iteration is stable so that the condition on the incremental change in filter coefficients to ensure stability can be imposed.

Our formulation is a modification of this method in that we first solve the problem in (3-4). If the given pole radius constraint is satisfied, then the solution obtained is the desired result. If not, the radii of those poles exceeding the prescribed pole radius will be reduced to obtain a feasible initial solution. The Rouche's theorem, which will be formulated as a set of LMIs in the sequel, will be invoked together with (3-4) to obtain the solution. To proceed further, let's start with the Rouche's theorem as follows:

Rouche's theorem [8,9]

If $f(z)$ and $g(z)$ are analytic inside and on a simple closed contour C, and if $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C. To enforce the stability constraints using the Rouche's theorem, consider the function

$$A(z^{-1}) = \sum_{n=0}^{N} a_n z^n = 1 + \sum_{n=1}^{N} a_n z^n . \qquad (3\text{-}6)$$

Note the denominator of the filter is given by $A(z)$. Hence, the zeros of $A(z^{-1})$ and $A(z)$ are reciprocals of each other.

Let the denominator of the all-pass filter obtained at the *i*-th iteration be $A^{(i)}(z) = 1 + a_1^{(i)} z^{-1} + \cdots + a_N^{(i)} z^{-N}$. We hope to solve for the coefficients in the following incremental polynomial

$$\Delta^{(i)}(z) = \Delta_0 + \Delta_1 z^{-1} + \cdots + \Delta_N z^{-N},\ n = 1,\cdots,N,$$

where $$A^{(i+1)}(z) = A^{(i)}(z) + \Delta^{(i)}(z) \qquad (3\text{-}7)$$

is the new denominator which has all its zero inside { C_ρ : $|z| = \rho < 1$ } while minimizing the minimax design criterion. Choosing the functions *f*(*z*) and *g*(*z*) as

$$g(z) = z^N \Delta^{(i)}(z) = \Delta_0 z^N + \Delta_1 z^{N-1} + \cdots + \Delta_N, \qquad (3\text{-}8)$$

$$f(z) = z^N A^{(i)}(z) = z^N + a_1^{(i)} z^{N-1} + \cdots + a_N^{(i)}, \qquad (3\text{-}9)$$

which are analytic inside and on C_ρ, it then follows from the Rouche's theorem that $f(z)+g(z)$ and $f(z)$ have the same numbers of zeros inside C_ρ if

$$|g(z)| < |f(z)|,\ |z| = \rho . \qquad (3\text{-}10)$$

By enforcing (3-10), $A^{(i+1)}(z)$ and $A^{(i)}(z)$ will have the same numbers of zeros inside C_ρ. If $A^{(i)}(z)$ has all its root inside C_ρ, then so is $A^{(i+1)}(z)$, and the desired pole radius constraint is imposed. (3-10) can also be written in terms of $\Delta^{(i)}(z)$ as

$$\left|\Delta^{(i)}(z)\right| < \left|A^{(i)}(z)\right|,\quad C_\rho : |z| = \rho < 1. \qquad (3\text{-}11)$$

Therefore, if the initial polynomial $A^{(0)}(z)$ is chosen with all its zeros inside C_ρ, then with constraint (3-11), all subsequent polynomials $A^{(i)}(z)$, $i = 0,1,\cdots$, will have their zeros inside C_ρ. Next, we shall reformulate (3-11) as a set of LMIs so that it can be solved using SDP. From (3-6) and (3-11), we have

$$\left|\sum_{n=0}^{N} \Delta_n \rho^{-n} e^{-jn\omega}\right|^2 < \left|\sum_{n=0}^{N} a_n^{(i)} \rho^{-n} e^{-jn\omega}\right|^2,\ -\pi \le \omega \le \pi$$

$$\Leftrightarrow \quad \chi^2 - (\beta_R^2 + \beta_I^2) > 0 \qquad (3\text{-}12)$$

Where $\chi^2 = (\boldsymbol{a}^{(i)T} \boldsymbol{c}_\rho)^2 + (\boldsymbol{a}^{(i)T} \boldsymbol{s}_\rho)^2$, $\boldsymbol{a}^{(i)} = [1\ \ a_1^{(i)}\ \cdots\ a_N^{(i)}]^T$,

$\beta_R = \boldsymbol{\Delta}^T \boldsymbol{c}_\rho$, $\beta_I = \boldsymbol{\Delta}^T \boldsymbol{s}_\rho$, $\boldsymbol{\Delta}^{(i)} = [\Delta_0\ \ \Delta_1\ \cdots\ \Delta_N]^T$;

$\boldsymbol{c}_\rho = [1\ \ \rho^{-1}\cos\omega\ \cdots\ \rho^{-N}\cos N\omega]^T$

and $\boldsymbol{s}_\rho = [0\ \ \rho^{-1}\sin\omega\ \cdots\ \rho^{-N}\sin N\omega]^T$.

(3-12) is equivalent to

$$\boldsymbol{\Lambda}(\boldsymbol{\Delta},\omega) = \begin{pmatrix} \chi^2 & \beta_R(\omega) & \beta_{Ii}(\omega) \\ \beta_R(\omega) & 1 & 0 \\ \beta_{Ii}(\omega) & 0 & 1 \end{pmatrix} > 0,\ -\pi \le \omega \le \pi . \qquad (3\text{-}13)$$

Discretize the frequency variable ω in the band of interest into L points, ω_l for $l = 1,\cdots,L$, we obtain a finite set of constraints $\boldsymbol{\Lambda}_l(\boldsymbol{\Delta},\omega_l) > 0$, for $l = 1,\cdots,L$. Stacking the constraints together, one obtains the discretized version of (3-13) as follows:

$$\boldsymbol{\Lambda}(\boldsymbol{\Delta}) = diag\{\boldsymbol{\Lambda}_1(\boldsymbol{\Delta}),\cdots,\boldsymbol{\Lambda}_L(\boldsymbol{\Delta})\} > 0 . \qquad (3\text{-}14)$$

We now modify $\boldsymbol{F}_k(\boldsymbol{a})$'s in (3-4) to incorporate these constraints. Substitute (3-6), (3-7) into (3-4) and expressing the variable $\boldsymbol{a}$ in terms of $\boldsymbol{\Delta}$ and $\boldsymbol{a}^{(i)}$, one gets the following for $\boldsymbol{F}_k$ at the $i+1$-th iteration:

$$\boldsymbol{F}_k(\boldsymbol{\Delta}) = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix},$$

$$v_1 = (\boldsymbol{a}^{(i)T} + \boldsymbol{\Delta}) \cdot (\delta \cdot \boldsymbol{c}(\omega_k) + W(\omega_k) \cdot \boldsymbol{s}(\omega_k)), \qquad (3\text{-}15)$$

$$v_2 = (\boldsymbol{a}^{(i)T} + \boldsymbol{\Delta}) \cdot (\delta \cdot \boldsymbol{c}(\omega_k) - W(\omega_k) \cdot \boldsymbol{s}(\omega_k)).$$

The feasibility problem to be solved is then given by:

$$\min_{\boldsymbol{\Delta}} \varepsilon \ \text{ subject to } diag\{\boldsymbol{F}(\boldsymbol{a},\varepsilon), \boldsymbol{\Lambda}(\boldsymbol{a})\} \ge 0 . \qquad (3\text{-}16)$$

Using this SDP and the bisection method, the all-pass filter with the minimum phase ripple and prescribed maximum pole radius can be obtained. Figure 1 summarizes the proposed design algorithm.

IV. DESIGN EXAMPLES

We illustrate the proposed design method by considering several design examples. The SDP was solved using the *Matlab LMI toolbox*. In the first example, an all-pass filter is designed to realize a digital halfband filter (HBF). The transfer function of the HBF is given by $H(z) = \frac{1}{2}(z^{-16} + z^{-1}\beta(z^2))$, where $\beta(z)$ is an all-pass digital filters. The desired response of $\beta(z)$ is given by $\beta_d(e^{j\omega}) = e^{-j(8-1/2)\omega}$, $\omega \in [0, 2\omega_p]$, where ω_p is the passband edge of the halfband filter. In the passband, i.e. $\omega \in [0,\omega_p]$, $e^{-j\omega}\beta(e^{j2\omega}))$ approximates $e^{-j16\omega}$, and hence $H(e^{j\omega})$ is approximately equal to $e^{-j16\omega}$. In the stopband, $\omega \in [\omega_s, \pi]$, $e^{-j\omega}\beta(e^{j2\omega}))$ approximates $-e^{-j16\omega}$, and hence $H(e^{j\omega})$ is close to zero. The order *N* of the allpass filter and ω_p in our design are set to 8 and $\omega_p = 0.4\pi$, respectively. This is identical to example 2 in [1] for comparison purpose. The frequency band of interest $\omega \in [0, 2\omega_p]$ is discretized uniformly to form *K*=700 constraints. The maximum pole radii ρ are set to 0.98, 0.8, and 0.78, and the frequency variable ω of the stability contour *C* in (3-7) is discretized uniformly into *L*=700 points. The tolerance of the ripple δ is set to 0.01, as a reasonable tradeoff between computational time and final accuracy. The frequency responses of the halfband filters obtained are shown in Figure 2(a). Their passband ripple and group delay responses are shown in Figures 2(b) and (c), respectively. The upper and lower bound values of the ripple error are 0.000238 and 0.000235, respectively. This difference can be made arbitrary small by reducing the tolerance of δ. Figure 2(d) shows the pole-zero plot of the allpass filter with ρ = 0.98. From these figures, we can see that the stopband is approximately equi-ripple with a highly linear phase characteristic in the passband and stopband. The stopband attenuation for ρ = 0.98 is 72.467dB, which is very close to 72.5dB that was obtained in example 2 of [1]. The latter was known to be the best result reported so far. From Figure 2(d), it can be seen that the outermost pole is bounded by specified pole radius. On the other hand, if the pole constraint is smaller than 0.81, the ripple will be increased to satisfy the smaller prescribed pole radius constraint. Since the stopband attenuation between successive values of N is rather large, for some applications, the stopband attenuation of the AP-based digital filters might exceed the given specification of stopband attenuation by a considerable margin. The stopband attenuation can be traded for better stability margin as illustrated in this example. Hence, the pole radius constraint allows an additional tradeoff between the approximation error and the stability margin in finite wordlength implementation. To demonstrate the usefulness of the proposed approach in imposing linear constraints, 3 and 15 zeros at $\omega = \pi$ are respectively imposed as linear constraints in (3-16). The frequency responses of the resulting HFBs are shown in figure 3. We have also done a comparison with the results in [3,7,14], and the results are either identical to or better than (in the multiband case [14]) the results reported. Due to page limitation, these results are omitted here.

V. CONCLUSION

A new method for designing digital all-pass filters with a minimax design criterion using SDP is presented. Unlike other all-pass filter design methods, additional linear constraints can be readily incorporated. The convergence of the algorithm is guaranteed. Nonlinear constraints such as the pole radius constraint and hence the stability margin of the filters can also be formulated as LMIs using the Rouche's theorem. This allows an additional tradeoff between the approximation error and the stability margin in finite wordlength implementation. The effectiveness of the proposed method is demonstrated by several design examples.

REFERENCES

[1] M. Lang, "All-pass filter design and applications," *IEEE Trans. Signal Processing*, vol.46, pp.2505-2514, Sept.1998.

[2] T. Q. Nguyen, T. I. Laakso and R. D. Koilpillai, "Eigenfilter approach for the design of all-pass filters approximating a given phase response," *IEEE Trans. Signal Processing*, vol. 42,pp. 2257-2263, Sept. 1994.

[3] Zhang Xi and H. Iwakura, "Design of IIR digital all-pass filters based on eigenvalue problem," *IEEE Trans. Signal Processing*, vol. 47, pp. 554-559, Feb 1999.

[4] P. A. Regalia, S. K. Mitra and P. P. Vaidyanathan, "The digital all-pass filter: a versatile signal processing building block," *Proc. IEEE*, vol. 76, pp. 19-37, Jan. 1988.

[5] C. K. Chen and J. H. Lee, "Design of digital all-pass filters using a weighted least squares approach," *IEEE Trans. Circuits Syst. II*, vol. 41, pp. 346-350, May 1994.

[6] S. C. Pei and J. J. Shyu, "Eigenfilter design of 1-D and 2-D IIR digital all-pass filters," *IEEE Trans. Signal Processing*, vol. 42, pp. 996-968, Apr. 1994.

[7] M. Ikehara, H. Tanaka and H. Kuroda, "Design of IIR digital filters using all-pass networks," *IEEE Trans. Circuits Syst. II*, vol. 41, pp. 231-235, Mar.1994.

[8] R. V. Churchill and J. W. Brown, *Complex Variables and Applications*. New York: McGraw-Hill, 1984.

[9] M. R. Spiegel. Theory and problems of complex variables. Schaum's outline series, McGraw-Hill International Book Company. 1981.

[10] A. G. Deczky, "Equiripple and minimax (Chebyshev) approximations for recursive digital filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-22, ppp. 98-111, 1974.

[11] W. S. Lu, "Design of stable minimax IIR digital filters using semidefinite programming," in *Proc. ISCAS'2000*, May 28-31, 2000, Geneva, Switzerland, pp. I-355-358.

[12] Z. Jing, "A new method for digital all-pass digital filter design," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1557-1564, 1987.

[13] M. L. Lang, "Least-Squares Design of IIR filters with Prescribed Magnitude and Phase Responses and a Pole Radius Constraint," *IEEE Trans. On Signal Processing*, vol. 48, Nov. 2000, pp. 3109-3121.

[14] T. Saramaki and M. Renfors, "A Remez-Type algorithm for designing digital filters composed of all-pass sections based on phase approximations", *Proc. IEEE ISCAS*, vol. 1, pp. 571-575, Aug. 1995.

Initialization: $\delta_L := 0$, $\delta_U := \delta_{Init}$, $\rho := \rho_{Init}$

1. Test the feasibility of (3-4) for δ_U.
   If infeasible, no good solution is found.
   If the maximum pole radius is greater than ρ
   {Replace the radius of those poles by ρ, if it is larger than ρ.
   Test the feasibility of (3-16) for δ_U.
   If infeasible, no good solution is found.}
2. If $|1-(\delta_U/\delta_L)| < tolerance$, report the most recent solution.
   Else {
   $\delta \leftarrow (\delta_L + \delta_U)/2$;
   If no pole radius constraint is violated so far
   {Test the feasibility of (3-4) for δ.
   If the maximum pole radius is greater than ρ
   {Replace the radius of those poles by ρ, if it is larger than ρ.
   Test the feasibility of (3-16) for δ.} }
   Else
   Test the feasibility of (3-16) for δ.
   If feasible
   $\delta_U \leftarrow \delta$.
   Else $\delta_L \leftarrow \delta$
   goto 2.}

Figure 1. Summary of the proposed design method.

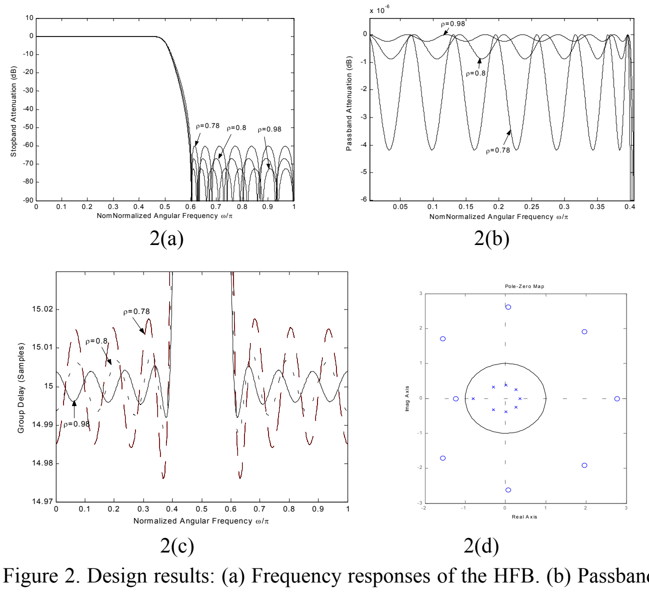

Figure 2. Design results: (a) Frequency responses of the HFB. (b) Passband ripple errors. (c) Group delay responses of the AP filter with ρ = 0.98, 0.8, and 0.78 (d) Pole-zero plot of the all-pass filters with ρ = 0.98. "O" and "X" stand for a zero and a pole respectively.

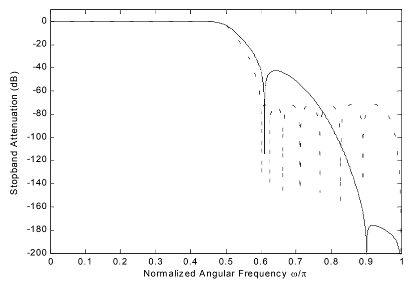

Figure 3. Magnitude responses of the AP-based HB filters: solid-line - 15 zeros at $\omega = \pi$, dotted-line - 3 zeros at $\omega = \pi$).