WEYL-HEISENBERG SIGNAL EXPANSIONS OVER R IN $l_2(\mathbf{Z})$ AND DUALITY RELATIONS INVOLVING MDFT FILTER BANKS

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ABSTRACT

Modified discrete Fourier transform (MDFT) filter banks are analyzed in relation with Weyl-Heisenberg expansions over $\mathbf{R}$ in $l_2(\mathbf{Z})$. This analysis is used to formally derive a result that could sometimes be taken for granted without any proof: the design of perfect reconstruction (PR) MDFT subband coders is equivalent to that of PR MDFT transmultiplexers. The framework of WH expansions over $\mathbf{R}$ in $l_2(\mathbf{Z})$ is also used to prove the equivalence between the MDFT transmultiplexer and the biorthogonal frequency division multiplex/offset quadrature amplitude modulation (BFDM/OQAM) multicarrier system. This analysis also leads to a slightly modified MDFT scheme with a reduced reconstruction delay.

1. INTRODUCTION

The filter bank theory is now widely used and has two main practical uses, which are subband coding and multicarrier modulation. It is well known that the design of both subband coders[1] and transmultiplexers with perfect reconstruction (PR) are two dual problems. It has already been shown that they are even equivalent in the case of an M-channel maximally decimated filter bank [1, 2] or in the case of oversampled DFT filter banks [3]. That is why it is often considered that this conclusion is also valid for any type of filter bank. But for instance in the case of a general oversampled filter bank the duality question is meaningless. The equivalence between PR subband coders and PR transmultiplexers mentioned above rests indeed on the fact that an M-channel maximally decimated filter bank is equivalent to a pair of biorthogonal bases over $\mathbf{C}$ in $l_2(\mathbf{Z})$, and that oversampled DFT filter banks are related to Weyl-Heisenberg (WH) expansions over $\mathbf{C}$ in $l_2(\mathbf{Z})$ [4]. But neither the first, nor the second argument applies to MDFT filter banks. And yet, it has recently been shown [5] that the BFDM/OQAM multicarrier modulation can be realized thanks to a transmultiplexer whose PR conditions are the same as the ones given for the MDFT subband coder [6]. Moreover, in spite of a few differences, this BFDM/OQAM transmultiplexer is very similar to the MDFT transmultiplexer used for the orthogonal multiple carrier data transmission (OMC) [7]. As in both cases the realization schemes involve taking the real part of the conventional inner product, an analysis based on WH expansions over $\mathbf{R}$ is now proposed.

Therefore, in this paper, we briefly present the framework of WH expansions over $\mathbf{R}$ in $l_2(\mathbf{Z})$. Then, MDFT filter banks are reviewed by separately considering their different subblocks. This leads us to an analysis of MDFT filter banks thanks to a special type of WH expansions over $\mathbf{R}$ in $l_2(\mathbf{Z})$.

2. WH EXPANSIONS OVER R IN $l_2(\mathbf{Z})$

The space of square summable sequences, $l_2(\mathbf{Z})$, is usually seen as a Hilbert space over $\mathbf{C}$ with the inner product $\langle u, v\rangle_{\mathbf{C}} = \sum_{k=-\infty}^{+\infty} u^*[k]v[k]$ and norm $\|u\| = \sqrt{\langle u, u\rangle_{\mathbf{C}}} = \sqrt{\sum_{k=-\infty}^{+\infty} |u[k]|^2}$. Thus, every $u \in l_2(\mathbf{Z})$ can be expanded as $u[k] = \sum_{n=-\infty}^{+\infty} u_n e_n[k]$, with $u_n \in \mathbf{C}$, provided that $(e_n)_{n\in\mathbf{Z}}$ spans $l_2(\mathbf{Z})$. But, it is clear that $l_2(\mathbf{Z})$ is also a vector space over $\mathbf{R}$ and even a Hilbert space over $\mathbf{R}$, with the inner product $\langle u, v\rangle_{\mathbf{R}} = \Re\left\{\sum_{k=-\infty}^{+\infty} u^*[k]v[k]\right\}$ and without changing the norm. Using these notations, it is possible to define the notions of frame and biorthogonality as in $l_2(\mathbf{Z})$ considered as a Hilbert space over $\mathbf{C}$ [8], except that we use a real-valued inner product. Thus, two sets of sequences $(u_n)_{n\in\mathbf{I}}$, $(v_n)_{n\in\mathbf{I}}$ are said to be biorthogonal when

$$\forall (n, n') \in \mathbf{I}^2,\ \langle u_n, v_{n'}\rangle_{\mathbf{R}} = \delta_{n,n'}, \tag{1}$$

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[1] Several authors refer to filter banks as subband coders, but here, as in [1], we do not make this restriction. We consider that subband coders and transmultiplexers constitute two different types of filter banks.

$\mathbf{I}$ being a set of indices and $\delta_{n,n'}$ being the Kronecker operator. Thus, if (u_n) and (v_n) are biorthogonal, and if $u = \sum_{n\in\mathbf{I}} a_n u_n$, then $a_n = \langle v_n, u\rangle_{\mathbf{R}}$.

Besides, $(u_n)_{n\in\mathbf{I}}$ is said to be a frame if there exists two finite, strictly positive real-valued numbers A, B, such that

$$\forall x \in l_2(\mathbf{Z}), A\|x\|^2 \leq \sum_{n\in\mathbf{I}} |\langle u_n, x\rangle_{\mathbf{R}}|^2 \leq B\|x\|^2, \quad (2)$$

and we say that two frames $(u_n)_{n\in\mathbf{I}}$, $(v_n)_{n\in\mathbf{I}}$ are dual if ([9])

$$\forall x \in l_2(\mathbf{Z}),\ x = \sum_{n\in\mathbf{I}} \langle u_n, x\rangle_{\mathbf{R}} v_n. \quad (3)$$

Let us now define WH systems in $l_2(\mathbf{Z})$ [9]. We say that $f \in l_2(\mathbf{Z})$ is the prototype function which defines the WH system $f_{m,n}$, $0 \leq m \leq N-1$, $n \in \mathbf{Z}$, with parameters M and N, $(M, N) \in \mathbf{N}^2$ if

$$f_{m,n}[k] = f[k - nM] e^{j\frac{2\pi}{N}mk}. \quad (4)$$

Thus, a WH expansion over $\mathbf{R}$, or over $\mathbf{C}$, is a signal which can be expanded as $\sum_{m=0}^{N-1}\sum_{n=-\infty}^{+\infty} a_{m,n} f_{m,n}$, with $a_{m,n} \in \mathbf{R}$, or $a_{m,n} \in \mathbf{C}$, respectively.

In the context of expansion over $\mathbf{R}$, it is useful to define a new type of WH systems, which we call Θ-WH systems, by

$$f^{\theta}_{m,n}[k] = f[k - nM] e^{j\frac{2\pi}{N}mk} e^{j\theta_{m,n}}, \quad (5)$$

with $\Theta : (m, n) \in [0, N-1] \times \mathbf{Z} \mapsto \theta_{m,n} \in \mathbf{R}$. In the case of expansions over $\mathbf{C}$, it can easily be shown that, $\forall \Theta, \Theta'$, $f^{\Theta}_{m,n}$ and $g^{\Theta}_{m,n}$ constitute two biorthogonal families or two dual frames if and only if $f^{\Theta'}_{m,n}$ and $g^{\Theta'}_{m,n}$ also constitute two biorthogonal families or two dual frames, respectively. But this is no longer true in the case of systems over $\mathbf{R}$. In this case, it can be shown that this equivalence is still true when $\exists \theta_0 \in \mathbf{R}$, such that $\theta'_{m,n} - \theta_{m,n} \equiv \theta_0 \pmod{\pi}$.

3. MDFT FILTER BANKS

MDFT filter banks (subband coders or transmultiplexers) can be built thanks to the same basic blocks depicted in figures 1, 2 and 3. The only difference between them is that the analysis (figure 4) is performed before the synthesis (figure 5) for the subband coder (see figure 6), whereas it is performed afterwards for the transmultiplexer (see figure 7). Let us write the input-output relations of these blocks:

- analysis filter bank (see figure 1)

$$y_m[n] = \sum_{k=-\infty}^{+\infty} h_m[nM - k] x[k], \quad (6)$$

$$\text{with } h_m[k] = h[k] e^{j\frac{2\pi}{2M}m(k-\frac{D}{2})}, \quad (7)$$

  h being an FIR filter and D being a strictly positive integer parameter;

- synthesis filter bank (see figure 2)

$$y[k] = \sum_{n=-\infty}^{+\infty}\sum_{m=0}^{2M-1} x_m[n] f_m[k - nM], \quad (8)$$

$$\text{with } f_m[k] = f[k] e^{j\frac{2\pi}{2M}m(k-\frac{D}{2})}, \quad (9)$$

  f being an FIR filter and D being the same strictly positive integer parameter as for the analysis;

- demultiplexer (see figure 3)

$$y^0_m[n] = y_m[2n], \quad y^1_m[n] = y_m[2n-1]; \quad (10)$$

- multiplexer (see figure 3)

$$x_m[2n+1] = x^0_m[n], \quad x_m[2n] = x^1_m[n]. \quad (11)$$

4. WH ANALYSIS OF MDFT FILTER BANKS

4.1. MDFT analysis filter bank

Let us first define the phase term ψ^i_m by $\psi^i_m = 0$ if $m+i$ is even, and $\psi^i_m = \frac{\pi}{2}$ if $m+i$ is odd. Then, using notations of figures 3 and 4, we have $\hat{y}^i_m[n] = \Re\left\{e^{-j\psi^i_m} y^i_m[n]\right\}$. Moreover, using equation (10) and denoting $\hat{y}_m[n] = \hat{y}^i_m[\frac{n+i}{2}]$ and $\varphi_{m,2n-i} = \psi^i_m$, we get $\hat{y}_m[n] = \Re\left\{e^{-j\varphi_{m,n}} y_m[n]\right\}$. Then, using equations (6) and (7), we finally obtain

$$\hat{y}_m[n] = \Re\left\{\sum_{k=-\infty}^{+\infty} h[nM-k] e^{j\frac{2\pi}{2M}m(nM-k-\frac{D}{2})} x[k] e^{-j\varphi_{m,n}}\right\}, = \langle g^{\Theta}_{m,n}[k], x[k-D]\rangle_{\mathbf{R}}, \quad (12)$$

with $g[k] = h^*[D-k]$, $\theta_{m,n} = \varphi_{m,n} - \frac{2\pi}{2M}m(nM + \frac{D}{2})$ and $N = 2M$. Thus, an MDFT analysis filter bank can be seen as an inner product in $l_2(\mathbf{Z})$ over $\mathbf{R}$ involving a Θ-WH system.

4.2. MDFT synthesis filter bank

Using notations of figures 3 and 5, we can also write that $x^i_m[n] = e^{j\psi^i_m}\hat{x}^i_m[n]$. Then, using equation (11) and denoting $\hat{x}_m[n] = \hat{x}^i_m[\frac{n+1-i}{2}]$, we get $x_m[n] = e^{j\varphi_{m,n-1}}\hat{x}_m[n]$. And, finally, using equations (8) and (9), we find that

$$y[k] = \sum_{m=0}^{2M-1}\sum_{n=-\infty}^{+\infty} f_m[k-nM] e^{j\varphi_{m,n-1}} \hat{x}_m[n], \quad (13)$$

$$y[k+M] = \sum_{m=0}^{2M-1}\sum_{n=-\infty}^{+\infty} f_m[k-nM] e^{j\varphi_{m,n}} \hat{x}_m[n+1], = \sum_{m=0}^{2M-1}\sum_{n=-\infty}^{+\infty} \hat{x}_m[n+1] f^{\Theta}_{m,n}[k]. \quad (14)$$

Moreover, in a back-to-back subband coder, $\hat{x}^{i}_{m}[n] = \hat{y}^{i}_{m}[n]$ so that we can guarantee that $\hat{x}_m[n] \in \mathbf{R}$. As for the MDFT transmultiplexer, we assume that its inputs $\hat{x}^{i}_{m}[n]$ are real-valued so that $\hat{x}_m[n] \in \mathbf{R}$. Therefore, an MDFT synthesis filter bank implements a Θ-WH expansion over $\mathbf{R}$.

4.3. Dual WH frames over R and the PR MDFT subband coder

An MDFT subband coder satisfies the PR property when there exists a strictly positive integer parameter k_d, such that $y[k] = x[k-k_d]$ under the assumption that $\hat{x}^{i}_{m}[n] = \hat{y}^{i}_{m}[n]$, $0 \le m \le 2M-1$ and $i \in \{0,1\}$, *i.e.* $\hat{x}_m[n] = \hat{y}_m[n-1]$. Moreover, we know from [6] that the reconstruction delay is necessarily $k_d = D + M$. Thus, using equations (12) and (14), we find that PR is equivalent to

$$x[k-D] = \sum_{m=0}^{2M-1} \sum_{n=-\infty}^{+\infty} \hat{y}_m[n] f^{\Theta}_{m,n}[k], \qquad (15)$$

$$\text{with } \hat{y}_m[n] = \langle g^{\Theta}_{m,n}[k], x[k-D]\rangle_{\mathbf{R}}, \qquad (16)$$

which is equivalent to the fact that $g^{\Theta}_{m,n}$ and $f^{\Theta}_{m,n}$ are two dual frames. It is worthwhile mentioning that, as already written in section 2, this is also equivalent to the fact that $g^{\Theta'}_{m,n}$ and $f^{\Theta'}_{m,n}$ are two dual frames, provided that $\theta_{m,n} - \theta'_{m,n} \equiv \theta_0 \pmod{\pi}$. Considering the definition of $\varphi_{m,n}$ in section 4.1, it appears that $\varphi_{m,n} \equiv \frac{\pi}{2}(m+n) \pmod{\pi}$. Thus, it is possible to define a generalized MDFT filter bank corresponding to $\varphi'_{m,n} \equiv \frac{\pi}{2}(m+n)+\theta_0 \pmod{\pi}$, satisfying the same PR conditions as the type I MDFT filter bank. For instance, it can be shown that the type II MDFT filter bank presented in [6] is another particular case of this generalized MDFT filter bank.

Besides, the multiplexing/demultiplexing steps, which allow us to alternately take the real part and the imaginary part of $x_m[n]$ during the analysis, also introduce a single sample delay, since $\hat{x}_m[n] = \hat{y}_m[n-1]$. Moreover, equation (14) shows that this one sample delay passed into the synthesis filter bank causes an M samples delay at the output. That is why it is possible to reduce by M samples the overall reconstruction delay if we directly put $\hat{y}_m[n]$ at the input of the synthesis filter bank, instead of $\hat{x}_m[n]$. Thus, if we avoid the multiplexing/demultiplexing middle stages, we can get a generalized MDFT subband coder with reduced delay, as depicted in figure 8.

4.4. Biorthogonal WH expansions over R and PR MDFT transmultiplexer

Let us now focus on the dual problem. We want to find f and g such that

$$\hat{y}_m[n] = \langle g^{\Theta}_{m,n}[k], x[k-D]\rangle_{\mathbf{R}} \text{ with} \qquad (17)$$

$$x[k-D] = \sum_{m=0}^{2M-1} \sum_{n=-\infty}^{+\infty} \hat{y}_m[n] f^{\Theta}_{m,n}[k], \qquad (18)$$

which means that $g^{\Theta}_{m,n}$ and $f^{\Theta}_{m,n}$ constitute a pair of biorthogonal families. One could think that this problem is exactly the same as the PR problem for an MDFT subband coder, but in fact the difference is that, in a subband coder, we know that the subband coefficients $\hat{y}_m[n]$ can be computed thanks to an analysis filter bank, whereas in this case, the coefficients $\hat{y}_m[n]$ can be any real-valued numbers and we have no guarantee that they can be computed by an analysis filter bank. Nevertheless, if we make this assumption, then, in order to get a delayed version $\hat{y}_m[n-\alpha]$ of $\hat{y}_m[n]$ at the output of the MDFT analysis filter bank, we have to put $x[k-\alpha M] = y[k+D-\alpha M]$, $y[k]$ being the signal output of the MDFT synthesis filter bank with input $\hat{y}_m[n]$. Thus, defining α and β by $D = \alpha M - \beta$, $0 \le \beta \le M-1$, we get $x[k-\alpha M] = y[k-\beta]$. In other words, when we know that $\hat{y}_m[n]$ can be computed thanks to an MDFT analysis filter bank, the PR for an MDFT subband coder is equivalent to the PR for the transmultiplexer depicted in figure 9, into which we have introduced a β samples delay along the transmission line.

We can see from [5] that this transmultiplexer corresponds to a BFDM/OQAM (or OFDM/OQAM [10]) modem. Thus, [6] and [5] show that the PR for an MDFT filter bank is in fact equivalent to the PR for this transmultiplexer, without any restriction on the real-valued symbols $\hat{y}_m[n]$. Thus, this proves that Θ-WH dual frames of $l_2(\mathbf{Z})$ over $\mathbf{R}$ are also in fact biorthogonal bases of $l_2(\mathbf{Z})$ over $\mathbf{R}$.

5. CONCLUSION

We have derived a new generalized MDFT subband coder with reduced delay, which is exactly the dual form of the transmultiplexer given in [5] and corresponds to a BFDM or OFDM/OQAM modem. We have also proved that PR for MDFT subband coders is related to a special type of dual WH frames in $l_2(\mathbf{Z})$ over $\mathbf{R}$ and that PR for the dual transmultiplexer is related to a biorthogonality condition for the same type of WH systems. Finally, using the PR conditions for MDFT subband coders and the BFDM/OQAM transmultiplexer given in [6] and [5] respectively, we have proved that, for this particular type of WH systems, the WH dual frames and the WH biorthogonal families are the same, and that both are pairs of biorthogonal bases of $l_2(\mathbf{Z})$ over $\mathbf{R}$.

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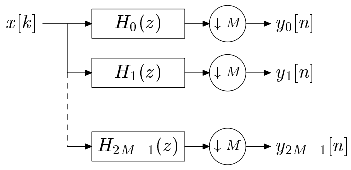

Fig. 1. Analysis filter bank (AFB).

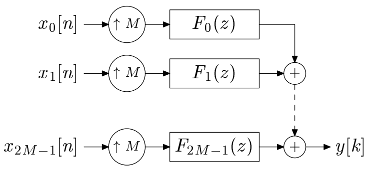

Fig. 2. Synthesis filter bank (SFB).

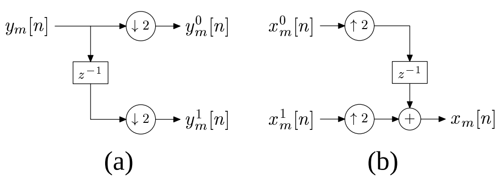

Fig. 3. (a) Demultiplexer (DMUX) and (b) Multiplexer (MUX).

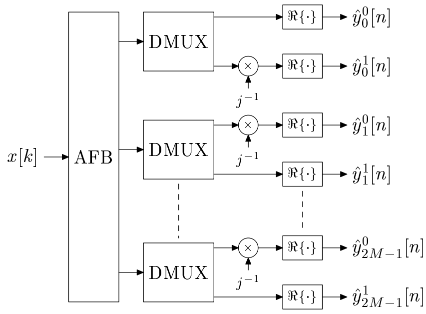

Fig. 4. MDFT analysis filter bank (MDFT-AFB).

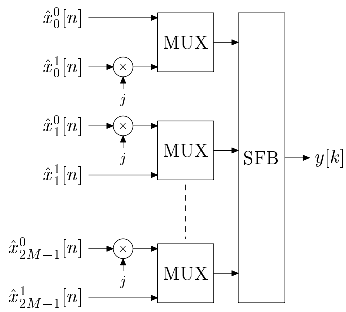

Fig. 5. MDFT synthesis filter bank (MDFT-SFB).

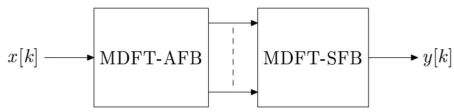

Fig. 6. MDFT subband coder.

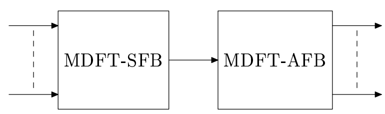

Fig. 7. MDFT transmultiplexer.

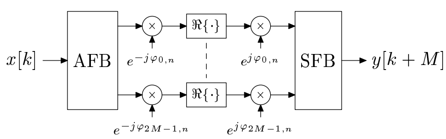

Fig. 8. Generalized MDFT subband coder with reduced delay.

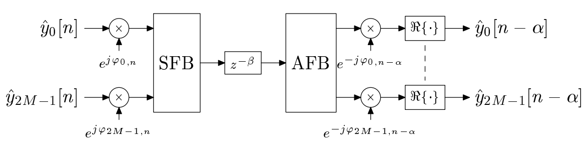

Fig. 9. BFDM/OQAM or OFDM/OQAM transmultiplexer.