CHARACTERIZATION OF APPROXIMATION ORDER OF MULTI-SCALING FUNCTIONS VIA REFINABLE SUPER FUNCTIONS

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ABSTRACT

In this work, we derive via refinable super functions characterization of approximation order of multi-scaling functions both in time and in frequency domains. It is shown that approximation order is achieved if the linear operator defined as the difference of the down-sampled convolution matrix and a matrix associated with the super function used has a zero eigenvalue. The left eigenvectors associated with the zero eigenvalue define the combinations of scaling functions that produce the desired refinable super function. In the frequency domain, the approximation order condition is expressed in terms of the refinement masks of the multi-scaling functions and the refinable super function. It shown that, implicit in this new characterization lies some well known results on approximation order. A matrix equality that equates the frequency characterization presented in this paper and Strang's well known characterization of accuracy is derived. It is shown that approximation order of multi-scaling functions can always be achieved by a refinable, compactly supported super function.

1. INTRODUCTION

Approximation order (p) is an important feature of all wavelets. It implies that polynomials up to degree $p-1$ are in the space spanned by the scaling function(s). In the scalar case, the scalar sum rules determine the approximation order. Heil, Strang and Strela [5], Plonka [8] and Jia [6] generalized these sum rules to multi-scaling functions. The results were expressed in terms of the existence of p non-zero vectors that satisfie a set of $2p$ equations. In [7], [8] and [9], Plonka *et. al.* studied the approximation properties of refinable function vectors in detail. It was shown that the factorization can be made almost identical to the well known factorization in the scalar case, i.e., the refinement mask can be factorized as the z-transform of a spline of order p times an arbitrary matrix polynomial.

In the seminal work of De Boor, DeVore and Ron [1], [2] [3], based on the super function theory, it was shown that in $L^2(R)$, the approximation order of a local (compactly supported) finitely generated shift invariant subspace can always be realized by one of its principal shift invariant subspaces spanned by a single function called the super function. In particular, the super function can always be selected to be compactly supported.

In this work, we characterize the approximation order of multi-scaling functions via *refinable* super functions. Specifically, we derive the necessary and sufficient conditions to characterize approximation order p both in time and in frequency domains. In the time domain, the characterization is formulated as an eigenvalue equation. In the frequency domain, the characterization takes a simple form which links the multi-scaling functions and the super function and allows us to establish the equivalence of our approximation order characterization with the well known characterization of accuracy [5],[8] and [6]. It is concluded that the approximation order of multi-scaling functions can always be achieved by compactly supported refinable super functions.

The rest of the paper is organized as follows. In section 2, results on approximation order of multi-scaling functions are recalled. First part of section 3 deals with the time domain characterization of approximation order based on the refinable super function. In the second part, frequency domain characterization is presented. In section 4 we establish the equivalence of our characterization and the well known characterization of accuracy.The conclusions are given in Section 5.

Notations: $\mathbf{Z}$ denotes the set of integers, $\mathbf{C}^r$ denote the r dimensional space of complex numbers, $\mathbf{C}^{r\times r}$ denotes the space of r by r matrices. For $X(\omega)$, the Fourier transform of x_k, X_0^j and X_π^j denote respectively the j^{th} ($j\geq 0$) derivative of $X(\omega)$ evaluated at $\omega=0$ and $\omega=\pi$, i.e., $X^{(j)}(0)$ and $X^{(j)}(\pi)$. In case $j=0$, we simply write X_0 and X_π. $\binom{j}{m}$ is the combination of m over j. I and I_2 are the identity matrices of size r and $2r$ respectively.

2. PRELIMINARIES

Multi-resolution can be generated not just in the scalar context, i.e., with one scaling function and one wavelet but in the vector case with r scaling functions and r wavelets as well. The latter leads to the notion of multi-wavelets. A multi-scaling function is defined by the dilation or refinement equation

$$\Phi(t)=\sum_{k\in\mathbf{Z}} C_k\Phi(2t-k) \tag{1}$$

where $C_k\in\mathbf{C}^{r\times r}$ and $\Phi(t)=[\phi_0(t)\ \phi_1(t)\ \phi_2(t)\ \ldots\ \phi_{r-1}(t)]^T$. It is known that C_k's uniquely define the scaling function $\Phi(t)$. Functions satisfying (1) are called refinable functions. If the summation is of finite terms, the scaling functions are compactly supported. If r=1, we have the scalar scaling function. We say that the multi-scaling function $\Phi(t)$ has approximation order $p(\geq 1)$ if polynomials of order up to $p-1$ lie in the linear span of integer translates of this scaling function $\Phi(t)$. That is, there exist

$y_k^j \in \mathbf{C}^r$ ($k \in \mathbf{Z}$, $k = 0, 1, \ldots p-1$) each of length r such that

$$\sum_{k \in \mathbf{Z}} y_k^j \Phi(t+n) = t^j, \quad j = 0, 1, \ldots p-1. \tag{2}$$

In order to achieve approximation order p in the scalar case it is necessary and sufficient to satisfy the following equivalent conditions.

$$\text{sum rules :} \quad \sum_{k \in \mathbf{Z}} (-1)^k k^j C_k = 0, \quad j = 0, 1, \ldots, \; p-1, \tag{3}$$

and

$$\text{zeros at } \pi : \; M(e^{i\omega}) = \frac{1}{2} \sum_{k \in \mathbf{Z}} C_k e^{-ik\omega} \texttt{ has } p \texttt{ zeros at } \pi \tag{4}$$

In the case of multi-scaling functions, Heil, Strang and Strela [5], Plonka [8] and Jia [6] gave the following characterization in terms of the matrix filter coefficients. Assume $\Phi(t)$ is an integrable solution of the matrix refinement equation such that the integer translates of $\phi_0, \phi_1, \ldots, \phi_{r-1}$ are independent. Then $\Phi(t)$ has accuracy p if and only if there are vectors $y^0, \, y^1, \, \ldots, \, y^{p-1} \in \mathbf{C}^r$ satisfying the two conditions in (5) and (6) for $j = 0, \, 1, \, \ldots, \, p-1$:

$$\sum_{m=0}^{j} \binom{j}{m} 2^m i^{j-m} y^m M^{(j-m)}(\pi) = 0 \tag{5}$$

$$\sum_{m=0}^{j} \binom{j}{m} 2^m i^{j-m} y^m M^{(j-m)}(0) = y^j. \tag{6}$$

3. CHARACTERIZATION OF APPROXIMATION ORDER VIA REFINABLE SUPER FUNCTIONS

Super functions possessing approximation order p can locally produce polynomials up to degree $p-1$. As shown in [1], [2] and [3], a super function can always be found with the same approximation order as the multi-scaling function. Thus it is natural to try to characterize the approximation order of multi-scaling functions via super functions that has the desired approximation order. In fact, the approximation order of one of the first multi-scaling functions was established by showing that the Hat function lies in the linear span of the integer translates of the multi-scaling functions [12], [4]. Motivated from this observation, we derive in the next two sub-sections time and frequency domain characterizations of approximation order via refinable super functions. We assume that the translates of the scaling functions $\Phi(t)$ are linearly independent.

3.1. Time domain characterization

It is required that linear combination of multi-scaling functions produces a given super function $f(t)$ with desired approximation order, i.e.,

$$f(t) = \sum_{\ell \in \mathbf{Z}} x_\ell \Phi(t-\ell) \tag{7}$$

where $x_\ell = [a_\ell^0 \;\; a_\ell^1 \;\; \ldots \;\; a_\ell^{r-1}]$.

If the super function $f(t)$ is refinable then there exists a sequence b_k such that

$$f(t) = \sum_{k \in \mathbf{Z}} b_k f(2t-k) \tag{8}$$

where b_k is the sequence that defines the refinable super function. Substituting (7) into the refinability requirement (8) gives

$$\sum_{\ell \in \mathbf{Z}} x_\ell \Phi(t-\ell) = \sum_{k \in \mathbf{Z}} b_k \left(\sum_{\ell \in \mathbf{Z}} x_\ell \Phi(2t-k-\ell) \right) \tag{9}$$

Substituting the matrix dilation equation (1) into the left hand side of (9) yields

$$\mathbf{x} L = \mathbf{x} B \tag{10}$$

or

$$\mathbf{x}\,(L-B) \, = \, 0 \tag{11}$$

where L is the down-sampled convolution matrix,

$$L = \begin{pmatrix} \ddots & & & & & & & \\ \ldots & C_{-1} & C_0 & C_1 & \ldots & & & \\ & & & C_{-1} & C_0 & C_1 & \ldots & \\ & & & & & C_{-1} & C_0 & \ldots \\ & & & & & & & \ddots \end{pmatrix} \tag{12}$$

B is a matrix defined by the super function

$$B = \begin{pmatrix} \ddots & & & & & & \\ \ldots & b_{-1}I & b_0 I & b_1 I & \ldots & & \\ & & b_{-1}I & b_0 I & b_1 I & \ldots & \\ & & & b_{-1}I & b_0 I & b_1 I & \ldots \\ & & & & & & \ddots \end{pmatrix} \tag{13}$$

and $\mathbf{x} = [\, \ldots \;\; x_{-1} \;\; x_0 \;\; x_1 \;\; \ldots \,]$.

Equation (10) or (11) characterizes approximation order of multi-scaling functions in the time domain. To summarize, the multi-scaling functions $\Phi(t)$ has approximation order p if and only if

a) the super function $f(t)$ defined by b_k has approximation order p, and

b) the matrix $L-B$ has a zero eigenvalue.

Condition a) can be satisfied if b_k is the regularity sequence convolved with any nonzero sequence. The regularity sequence is defined by the coefficients of z^k (for $k = 0, 1, \ldots p$) of the polynomial $(z+1)^p$. In other words, it is a sequence coming from Pascal's triangle which guarantees p zeros at z = -1 in the z-transform $B(z)$ of b_k. It is noted that the coefficients used in the linear combination to construct the super function $f(t)$ from the multi-scaling functions constitute the left eigenvector associated with the zero eigenvalue of $L-B$. In other words, any vector in the left null space of $L-B$ will produce the super function through equation (7). This is similar to the corresponding result in the scalar wavelets [11] where the left eigenvectors of the infinite down-sampled convolution matrix L describe the combinations of scaling functions for which t^j $(j = 0, 1, \; \ldots \; p-1)$ can be constructed.

3.2. Frequency domain characterization

It is an easy matter to verify that (10) is equivalent to the following matrix equation.

$$\mathbf{x}^u \, L^u = \mathbf{x} \, B \tag{14}$$

where L^u is the convolution matrix

$$L^u = \begin{pmatrix} \ddots & & & & & & \\ \dots & C_{-1} & C_0 & C_1 & \dots & & \\ & & C_{-1} & C_0 & C_1 & \dots & \\ & & & C_{-1} & C_0 & C_1 & \dots \\ & & & & & & \ddots \end{pmatrix} \tag{15}$$

and $\mathbf{x}^u = [\dots \;\; x_{-1} \;\; 0 \;\; x_0 \;\; 0 \;\; x_1 \;\; \dots \;\;]$.

Taking the discrete time Fourier Transform of (14), one obtains

$$X(2\omega)M(\omega) = X(\omega)B(\omega) \tag{16}$$

where

$$B(\omega) = \hat{b}(\omega)I, \quad \hat{b}(\omega) = \sum_{k\in\mathbf{Z}} b_k e^{-ik\omega} \tag{17}$$

$$X(\omega) = \sum_{k\in\mathbf{Z}} x_k e^{-ik\omega} \tag{18}$$

are the discrete time Fourier Transforms of the coefficients b_k defining the super function and the coefficients producing the super function respectively. The frequency domain characterization of the approximation order is stated as follows. The multi-scaling function $\Phi(t)$ has approximation order p if and only if

a) $\hat{b}(\omega)$ has p zeros at $\omega = \pi$; and

b) $X(2\omega)M(\omega) - X(\omega)B(\omega) = 0$

Again, condition a) is achieved by the regularity sequence b_k. Condition b) links the super function with the multi-scaling functions in the frequency domain. It is this frequency domain linkage that guarantees the approximation order of the super function is passed to the multi-scaling functions. For the case where the multi-scaling functions are compactly supported, it can be shown that condition b) can be simplified as some rank conditions. This will be studied in the next section.

4. EQUIVALENCE OF THE TWO CHARACTERIZATIONS

First we write equations (5) and (6) in the following equivalent form

$$\begin{bmatrix} i^j y^0 & i^{j-1} y^1 & \dots & i^0 y^j \end{bmatrix} \begin{pmatrix} 2^0 \binom{j}{0} M_\pi^j \\ 2^1 \binom{j}{1} M_\pi^{j-1} \\ \vdots \\ 2^j \binom{j}{j} M_\pi \end{pmatrix} = 0 \tag{19}$$

and

$$\begin{bmatrix} i^j y^0 & i^{j-1} y^1 & \dots & i^0 y^j \end{bmatrix} \begin{pmatrix} 2^0 \binom{j}{0} M_0^j \\ 2^1 \binom{j}{1} M_0^{j-1} \\ \vdots \\ 2^j \binom{j}{j} M_0 \end{pmatrix} = 0 \tag{20}$$

It follows that the existence of $\begin{bmatrix} y^0 \; y^1 \; \dots \; y^{p-1} \end{bmatrix} \in \mathbf{C}^r$ satisfying (5) and (6) are equivalent to the following rank conditions

$$\text{rank} \begin{bmatrix} M_S^j \end{bmatrix} < (j+1)\, r \quad j = 0, 1, \, \dots \, p-1 \tag{21}$$

where

$$M_S^j = \begin{pmatrix} M_\pi & M_0 - I & \dots & 2^0 \binom{j}{0} M_\pi^j & 2^0 \binom{j}{0} M_0^j \\ 0 & 0 & \dots & 2^1 \binom{j}{1} M_\pi^{j-1} & 2^1 \binom{j}{1} M_0^{j-1} \\ 0 & 0 & \dots & 2^2 \binom{j}{2} M_\pi^{j-2} & 2^2 \binom{j}{2} M_0^{j-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2^j \binom{j}{j} M_\pi & 2^j \binom{j}{j} M_0 - I \end{pmatrix} \tag{22}$$

In other words the scaling function $\Phi(t)$ has approximation order p if and only if none of the first $(j+1)r$ by $2(j+1)r$ ($j = 0, 1, \dots p-1$) block matrices are of full row rank.

We now consider the frequency domain characterization b). Evaluating the expression at $\omega = 0$ and $\omega = \pi$ and noting the 2π periodicity of the discrete time Fourier Transform, one obtains

$$X_0\, M_0 - X_0\, B_0 = 0 \quad \text{and} \quad X_0\, M_\pi - X_\pi B_\pi = 0 \tag{23}$$

If the super function has approximation order p then $B_\pi = B_\pi^1 = \dots = B_\pi^{p-1} = 0$ and we assume without loss of generality that $B_0 = 1$. Hence

$$X_0\; (M_0 - B_0) = 0 \quad \text{and} \quad X_0\, M_\pi = 0 \tag{24}$$

The first derivative of the frequency domain expression gives

$$2X'(2\omega)M(\omega) + X(2\omega)M'(\omega) = X'(\omega)B(\omega) + X(\omega)B'(\omega) \tag{25}$$

when evaluated at $\omega = 0$ and $\omega = \pi$, we have

$$X_0(M_0^1 - B_0^1) \; + \; X_0^1(2M_0 - B_0) \; = \; 0 \tag{26}$$

$$X_0 M_\pi^1 \; + \; 2\, X_0^1 M_\pi \; = \; 0 \tag{27}$$

Proceeding to take the j^{th} ($j < p$) derivative, we obtain the following matrix equations

$$\begin{bmatrix} X_0 & X_0^1 & \dots & X_0^j \end{bmatrix} M_B^j \; = \; 0, \tag{28}$$

where M_B^j is given by

$$\begin{pmatrix} M_\pi & M_0 - I & \dots & 2^0 \binom{j}{0} M_\pi^j & 2^0 \binom{j}{0} M_0^j - \binom{j}{0} B_0^j \\ 0 & 0 & \dots & 2^1 \binom{j}{1} M_\pi^{j-1} & 2^1 \binom{j}{1} M_0^{j-1} - \binom{j}{1} B_0^{j-1} \\ 0 & 0 & \dots & 2^2 \binom{j}{2} M_\pi^{j-2} & 2^2 \binom{j}{2} M_0^{j-2} - \binom{j}{2} B_0^{j-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2^{p-1} \binom{j}{j} M_\pi & 2^{p-1} \binom{j}{j} M_0 - I \end{pmatrix} \tag{29}$$

If $\Phi(t)$ is a solution of the matrix dilation equation with approximation order p than the matrix M_B^j is not of full row rank. The vectors in the left null space of M_B^j give constraints on the behavior of $X(\omega)$ at $\omega = 0$. Noting the similarities between the matrices given in (22) and (29) and the fact that they both are related to the problem of characterization of approximation order, one wonders how the two matrices M_B^j and M_B^j are related. With some matrix manipulations we establish the following matrix equality.

$$\Lambda_1\, M_S^j \; = \; M_B^j \Lambda_2 \tag{30}$$

where

$$\Lambda_1 = \begin{pmatrix} I & \lambda_1 I & \lambda_2 I & \dots & \binom{j}{0} \lambda_j I \\ 0 & I & 2\lambda_1 I & \dots & \binom{j}{1} \lambda_j I \\ 0 & 0 & I & \dots & \binom{j}{2} \lambda_j I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \tag{31}$$

$$\Lambda_2 = \begin{pmatrix} I_2 & \lambda_1 I_2 & \lambda_2 I_2 & \dots & 2^j \binom{j}{0} \lambda_j I_2 \\ 0 & I_2 & 2\lambda_1 I_2 & \dots & 2^{j-1}\binom{j}{1}\lambda_{j-1} I_2 \\ 0 & 0 & I_2 & \dots & 2^{j-2}\binom{j}{2}\lambda_{j-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & I_2 \end{pmatrix} \tag{32}$$

with

$$\lambda_1 = -B_0^1 \tag{33}$$

$$\lambda_2 = -\frac{1}{3}B_0^2 - \frac{4}{3}B_0^1\lambda_1 \tag{34}$$

and in general

$$\lambda_j = -\sum_{m=0}^{j-1} \frac{2^m \binom{j}{j-m}}{2^j - 1} B_0^{j-m} \lambda_m, \quad j \leq p \tag{35}$$

with $\lambda_0 = 1$.

Since both Λ_1 and Λ_2 are non-singular matrices, (30) implies that M_S^j and M_B^j have the same rank. Consequently the rank conditions (21) for accuracy p is equivalent to the following rank conditions

$$\text{rank}\left[M_B^j\right] < (j+1)\,r \quad j = 0, 1, \dots p-1. \tag{36}$$

This again is equivalent to the existence of non-zero $[X_0 \;\; X_0^1 \;\dots\; X_0^j]$ satisfying (28). From the frequency condition b), it is obvious that the rank conditions (36) are necessary for approximation order p via refinable super functions. If furthermore, $X(\omega)$ can be uniquely determined by $[X_0 \;\; X_0^1 \;\dots\; X_0^{p-1}]$ which is possible in the case when $\mathbf{x}$ is of finite length less than p, the rank conditions are also sufficient. From (1), it follows that $X(\omega)$ will be of finite length if the super function $f(t)$ and multi-scaling function $\Phi(t)$ are compactly supported.

Therefore we can conclude that approximation order p can always be achieved via compactly supported refinable super functions.

Finally, it is noted that in the context of scalar scaling functions ($r = 1$), it can be verified that the rank conditions (36) implies

$$M_0 = 1 \quad \text{and} \quad M_\pi^j = 0\;, \quad j = 0, 1, \dots, p-1 \tag{37}$$

That is the sum rules are included in the rank conditions.

Also it follows from the matrix equality (30) that the vector coefficients in [5] are related with the X_0^j's by

$$[y^0 \;\; y^1 \;\; \dots \;\; y^{p-1}] = [X_0 \;\; X_0^1 \;\dots\; X_0^{p-1}]\,\Lambda_1^p\,\Lambda_i^{-1} \tag{38}$$

where

$$\Lambda_i = \text{diagblock}(i^{-(p-1)}I \;\; \dots \;\; i^{-1}I \;\; I) \tag{39}$$

5. CONCLUSION

Refinable super functions are used to characterize approximation order of multi-scaling functions both in time and in frequency domains. In the time domain, it is shown that approximation order is implied by the existence of a zero eigenvalue of a linear operator defined as the difference of the down-sampled convolution matrix and a matrix associated with the super function selected. The left eigenvectors associated with the zero eigenvalue determine the vector coefficient in the linear combination of scaling functions that produce the desired refinable super function. In the frequency domain, the approximation order condition is expressed in terms of the refinement masks of the multi-scaling functions and the refinable super function. A matrix equality that relates the frequency characterization presented in this paper and the well known characterization of accuracy is derived. Based on this equality, it is shown that approximation order of multi-scaling functions can always be achieved by a refinable, compactly supported super function.

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