GROUP DELAY APPROXIMATION OF ALLPASS DIGITAL FILTERS BY TRANSFORMING THE DESIRED RESPONSE

Tatsuya Matsunaga and Masaaki Ikehara

Dept. of EEE, Keio University
3-14-1 Hiyoshi Kohoku Yokohama, 223 Japan

ABSTRACT

In this paper, we present a new design method of allpass digital filters with equiripple group delay response. This method is based on solving a least squares solution iteratively. At each iteration, the desired group delay response is transformed so as to have equiripple error. By this method, an equiripple solution is obtained very quickly with less computational complexity

1. INTRODUCTION

The design of allpass digital filters with a prescribed group delay response is important in digital signal processing. A characteristic of allpass filters is that amplitude response is constant in all frequency axis and only the phase response changes. Therefore, the allpass filters have been used as phase and group delay equalizers. The compensation problem of the group delay response has an important effect on signal processing and transmission. The group delay equalizer is realized through a stable allpass filter.

However, since the approximation of the group delay response is formulated as a nonlinear problem, it is very difficult to approximate the group delay response directly. Instead of approximating the group delay, a lot of methods to approximate the phase response have been proposed[3]-[7]. However, these methods does not evaluate the group delay directly.

Deczky[1][2] have solved the nonlinear problem for group delay approximation by using Golden section and Newton's algorithm. However, the shortfalls of this algorithm include the need of initial values of filter coefficients and heavy computation, especially for higher-order filter design.

Indirect methods[8],[9] which approximate a magnitude response or phase response instead of a group delay response have been proposed. Yegnanarayana transformed the group delay response into the amplitude response by using a complex cepstram. This method has an advantage in that the computational complexity is low. However, it does not evaluate group delay error. Moreover, group delay error increases as the filter order increases.

In this paper, we propose a direct approximation of the group delay response. This method is based on the least squares method, and an optimal solution in the least squares sense can be obtained. Moreover, by transforming the desired group delay response iteratively, we can obtain equiripple solutions very quickly. Although equiripple solutions to group delay approximation problems are not necessarily optimal in the Chebyshev sence, it is true that equiripple solutions often give quite satisfactory results[1]. Then we consider such cases and show that the best uniform approximation can be obtained.

The contents of this paper are as follows. In Section2, the group delay response of allpass filters and the least squares method is described. Section3 contains an explanation of the transformation of the desired group delay response, which is the focus of this paper. In Section4, some examples are shown by computer simulation. Section 5 is the conclusion.

2. LEAST SQUARES METHOD

A. Group Delay Response of Allpass Filters

The frequency response of allpass filter with order N is expressed by

$$A(z) = \frac{z^{-N}P(z^{-1})}{P(z)}$$
$$(P(z) = \sum_{n=0}^{N} a_n z^{-n}, \quad a_0 = 1) \tag{1}$$

The group delay response of $A(z)|_{z=e^{j\omega}}$ is given by

$$\tau_A(\omega) = N - 2\tau_P(\omega)$$
$$(\tau_P(\omega) = -\frac{d\theta_P(\omega)}{d\omega}) \tag{2}$$

where $\tau_P(\omega)$ is the group delay of the denominator polynimial.

When the desired group delay response of the filter is assumed to be $\tau(\omega)$, the mean square error is defined as follows.

$$E_1 = \frac{1}{\pi}\int_0^{\pi}(\tau_A(\omega) - \tau_d(\omega))^2 d\omega \tag{3}$$

Instead of approximating the group delay of an allpass filter, it is easy to approximate the group delay of the denominator polynomial. Then $\hat{\tau}_d(\omega)$ is defined as follows:

$$\hat{\tau}_d(\omega) = (N - \tau_d(\omega))/2 \tag{4}$$

From eq.(2) and eq.(4), the minimization of E_1 is equivalent to the minimization of the following error function:

$$E_2 = \frac{1}{\pi}\int_0^{\pi}(\tau_P(\omega) - \hat{\tau}_d(\omega))^2 d\omega \tag{5}$$

Therefore, we will solve the group delay approximation problem of the denominator polynomial.

B. Least Squares Method

The group delay response $\tau_P(\omega)$ of the denominator polynomial is expressed by

$$\tau_P(\omega) = Re\left[z\frac{dP(z)/dz}{P(z)}\right]_{z=e^{j\omega}} \tag{6}$$

where

$$[zdP(z)/dz]_{z=e^{j\omega}} = \hat{P}(\omega) = \sum_{n=1}^{N} na_n e^{-jn\omega} \tag{7}$$

Also, we denote $P(z)$ as

$$P(e^{j\omega}) = |P(e^{j\omega})| e^{j\theta_P(\omega)} \tag{8}$$

The group delay response of the denominator polynomial is written by

$$\begin{aligned} \tau_P(e^{j\omega}) &= Re\{\frac{1}{|P(e^{j\omega})|} e^{-j\theta_P(\omega)} \sum_{n=1}^{N} na_n e^{-jn\omega}\} \\ &= \frac{1}{|P(e^{j\omega})|} \sum_{n=1}^{N} na_n cos(n\omega + \theta_P(\omega)) \end{aligned} \tag{9}$$

Using discrete frequency points and matrix representation, the error function is written as

$$\mathbf{E} = \mathbf{Q}\mathbf{a} - \mathbf{T} \tag{10}$$

where

$$Q = \begin{bmatrix} \frac{cos(1\cdot\omega_0+\theta_P(\omega_0))}{|P(e^{j\omega_0})|} & \cdots & \frac{cos(N\cdot\omega_0+\theta_P(\omega_0))}{|P(\omega_0)|} \\ \frac{cos(1\cdot\omega_1+\theta_P(\omega_1))}{|P(e^{j\omega_1})|} & \cdots & \frac{cos(N\cdot\omega_1+\theta_P(\omega_1))}{|P(e^{j\omega_1})|} \\ \vdots & \ddots & \vdots \\ \frac{cos(1\cdot\omega_0+\theta_P(\omega_0))}{|P(e^{j\omega_{L-1}})|} & \cdots & \frac{cos(N\cdot\omega_0+\theta_P(\omega_0))}{|P(e^{j\omega_{L-1}})|} \end{bmatrix}$$

$$\mathbf{a} = [a_1 a_2 \cdots a_N]^T$$

$$\mathbf{T} = [\hat{\tau}_d(\omega_0)\hat{\tau}_d(\omega_1)\cdots\hat{\tau}_d(\omega_{L-1})]$$

Therefore, eq.(5) is rewritten by

$$\begin{aligned} E &= \mathbf{E}^T\mathbf{E} \\ &= \mathbf{a}^T\mathbf{Q}^T\mathbf{Q}\mathbf{a} - 2\mathbf{a}^T\mathbf{Q}^T\mathbf{T} + \mathbf{T}^T\mathbf{T} \end{aligned} \tag{11}$$

In order to minimize the eq.(11), E is differentiated by $\mathbf{a}$.

$$\frac{\partial E}{\partial \mathbf{a}} = 2\mathbf{Q}^T\mathbf{Q}\mathbf{a} - 2\mathbf{Q}^T\mathbf{T} = 0 \tag{12}$$

Therefore, **a** is expressed by

$$\mathbf{a} = (\mathbf{Q}^T\mathbf{Q})^{-1}\mathbf{Q}^T\mathbf{T} \tag{13}$$

Since we can define $a_0 = 1$ without loss of generality, we can obtain $\{a_n\}_{n=0}^{N}$ uniquely. However,the solution of eq.(13) is not an optimal solution of the group delay in a least squares sense because we take no account of $|P(e^{j\omega})|$ in eq.(9). Then, the obtained coefficients are used to calculate $P(e^{j\omega})$ and $P(e^{j\omega})$ is substituted into eq.(10) in the next iteration. As a result, this algorithm must be iterated until $P(z)$ has no change. If $\mathbf{a}$ converges, $\mathbf{a}$ is clearly an optimal solution in a least squares sense. Then we show the design algorithm as follows.

[**DESIGN ALGORITHM 1**]

1. Decide N and desired group delay response $\tau_d(\omega)$
2. set $P^0(\omega) = 1, \theta_P^0(\omega) = 0$
3. The filter coefficients are obtained by solving the eq.(13)
4. if $|P^{k-1}(\omega) - P^k(\omega)|/P^k(\omega) << 1$,terminate
5. Let $P^{k+1}(\omega) = P^k(\omega), \theta_P^{k+1}(\omega) = \theta_P^k(\omega)$, and go to step3

In this algorithm, the important point is that we start with $P(\omega) = 1$, $\theta_P(\omega) = 0$ and continue with $|P(\omega)|$, $\theta_P(\omega)$ calculated by an obtained set of coefficients on the previous iteration. Although we can not show enough proof of convergence, we have confirmed , through considerable experiences, this algorithm shows good convergence. A least squares solution is used as initial guess and is solved to get a equiripple solution in each iteration as shown in the next section.

3. TRANSFORMING THE DESIRED RESPONSE

As mentioned above, a least squares solution of the group delay is obtained very quickly. However, a least squares solution is not optimal in the minimax sense. In order to minimize the maximum value of the group delay error between the designed and desired response, an equiripple group delay error is desired. Although the equiripple solution may not be optimal in the minimax sense, it is true that equiripple solutions often give quite satisfactory results[1].

A basic idea to obtain the equiripple solution is to solve a least squares solution iteratively while changing the desired response such that the group delay error between the designed and the desired response becomes equiripple. First, we design an allpass filter based on a least squares method and find the local points of error between the desired and the designed response:

$$E_m(\omega) = \tau_m(\omega) - \tau_0(\omega) \tag{14}$$

where $\tau_m(\omega)$ is the group delay response derived by the least squares method in the previous section, $\tau_0(\omega)$ is the desired group delay response sqecified at the beginning of the algorithm, $E_m(\omega)$ is the error function and m is the number of iterations. We search for the points of local maximums of $|E_m(\omega)|$ as shown in Figure 1.

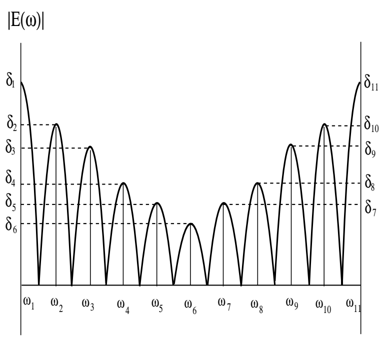

Fig. 1. Maximum points of Error $|E(\omega)|$

In Figure 1, ω_k is the local maximum points, δ_m^k is errors at the points of ω_k and $\delta_m^k = |E_m(\omega_k)|$. Next, we derive a new function $R_m(\omega_k)$ by transforming the error function $E_m(\omega_k)$ as follows:

$$R_m(\omega_k) = E_m(\omega_k)\frac{\delta_m}{\delta_m^k} \tag{15}$$

where δ_m is the average of δ_m^k, and

$$\delta_m = \frac{\sum_{k=1}^{M}\delta_m^k}{M}, \tag{16}$$

where M is the number of local maximum points. This corresponds to the scaling of the error function, as shown in Figure 2.

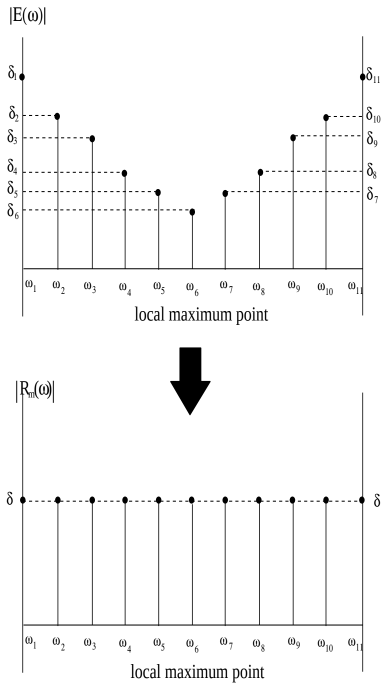

Fig. 2. Scaled Error Function $\mathbf{R}_m$

By adding the above error function $R_m(\omega_k)$ to the original desired function $\tau_0(\omega_k)$, the new desired function with an equiripple group delay error response is obtained:

$$\tau_d(\omega_k) = R_m(\omega_k) + \tau_0(\omega_k) \tag{17}$$

With this new desired response, a least squares solution is solved iteratively only on the local points. Because the number of local points is much less than the number of whole sampling points, the computational complexity of (13) is greatly reduced. Finally, we conclude the proposed algorithm as follows:

[PROPOSED ALGORITHM 2]

1. Calculate $\mathbf{a}$ by Algorithm1.
2. Calculate the group delay error by $E_m(\omega) = \tau_m(\omega) - \tau_0(\omega)$ (where m is the number of iterations)
3. . If $\frac{max\{|E_m(\omega)|\} - max\{|E_{m-1}(\omega)|\}}{max\{|E_m(\omega)|\}} < \epsilon\ (\epsilon \ll 1)$,then terminate. Otherwise, go to next step.
4. Calculate the new desired group delay response from eq.(15), (16) and (17).
5. With the new desired group delay response, a least squares solution is solved only on the local points by Algorithm1. Then a new $\tau_m(\omega)$ is derived.
6. Increase m, and go to Step 2.

At all iteration in Algorithm 1 and 2, $|P(\omega)|$ and $\theta_P(\omega)$ are caluculated by an obtained set of coefficients on the previous iteration. Although we can not show enough proof of convergence, since this algorithm is similar to Remez algorithm used in FIR filter design, a equiripple solution can be obtained with a few iterations. In practice we have confirmed, through considerable experiences, this algorithm shows good convergence.

4. DESIGN EXAMPLE

In this section, we show several examples of allpass digital filter to demonstrate the effectiveness of the proposed method.

4.1. Example 1

The specifications of the filter is as follows:

$$\begin{aligned} N &= 16 \\ \tau_d(\omega) &= \frac{16}{\pi}\omega + 7.974 \quad 0.1\pi \le \omega \le 0.99\pi \end{aligned}$$

This specification is the same as [1] and the error is defined as

$$E(\omega) = (\tau(\omega) - \tau_d(\omega))/(\frac{16}{\pi}\omega)$$

In this case, the error function in (14) is expressed by the above and (17) is rewritten by

$$\tau_d(\omega_k) = R_m(\omega_k)\frac{16}{\pi}\omega_k + \tau_0(\omega_k)$$

With this modification, the algorithm converges very quickly as shown in Fig.3. This algorithm needs 13 iterations and it took about 0.6 seconds to converge. Fig.3 shows the comparison of the group delay error between the proposed method and [1]. The maximum errors are 3.0427×10^{-3} in the proposed method and 3.1398×10^{-3} in [1], respectively.

4.2. Example 2

The specifications of the filter is as follows:

$$\begin{aligned} N &= 10 \\ \tau_d(\omega) &= \begin{cases} 13 & 0 \le \omega \le 0.3\pi \\ 7 & 0.6\pi \le \omega \le \pi \end{cases} \end{aligned}$$

Fig.4 shows the group delay response designed by this algorithm and the error between the designed and the desired response.

To compare this method with other methods, we transform the desired group delay into the desired phase response as follows:

$$\theta_d(\omega) = \begin{cases} -13\omega & 0 \le \omega \le 0.3\pi \\ -7\omega - 3\pi & 0.6\pi \le \omega \le \pi \end{cases}$$

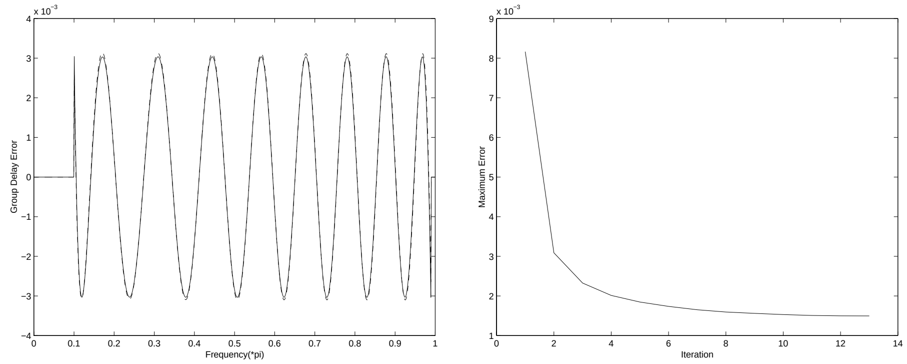

Fig. 3. Example 1: left:Group Delay Error of the proposed method(solid line) and [1](dashed line) right: Convergence

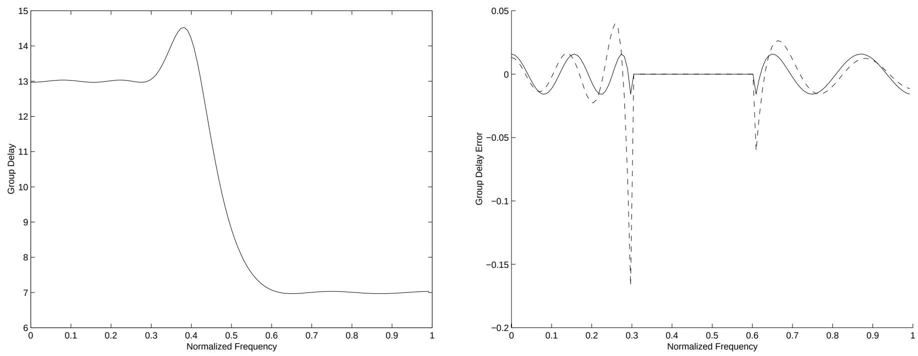

Fig. 4. Example 2, left:Group Delay Reponse, right:Group Delay error of the proposed method(solid line) and [7](dashed line)

Then we approximate the phase response by [7] such that the phase error becomes equiripple. The dashed line shows the group delay error, and the solid line shows the results obtained by the proposed method. As shown in these figures, the algorithm converges very quickly and can obtain the equiripple group delay response.

5. CONCLUSION

In this paper, we proposed a new design method of an all-pass digital filters using a least squares method by transforming the desired frequency response to obtain the equiripple solution. As a result, we could obtain equiripple solution in various specifications and degrees. Since this algorithm solves the linear equation iteratively only on local points without initial guess and solvin any nonlinear problems, its computational complexity is less than the conventional method. Also the algorithm converges with a few iteration because it is based on Remez algorithm.

6. REFERENCES

[1] A.G.Deckzky: "Equiripple and Minimax(Chebyshev) approximations for recursive digital filters", IEEE Trans. Acoust., Speech & Signal Processing, **ASSP-22**, No.2, pp.98-111, 1974

[2] A.G.Deczky: "Synthesis of Recursive Digital Filters Using the Minimum p-Error Criterion,"*IEEE Trans. Audio Electroacoust*,Vol.**AU-20**,pp.257-263,1972.

[3] Z.Jing: "A New Method for Digital Allpass Filter Design," *IEEE Trans. ASSP*,Vol.**ASSP-35**,pp.1557-1564,1987.

[4] X.Zhang and H.Iwakura: "Novel Method for Desiging Digital Allpass Filters Based on Eigenvalue Problem ",Electoronics Letters,Vol.29,Issue 14,8 July, pp.1279-1281,1993.

[5] M.Lang and T.I.Laakso: "Simple and Robust Method for the Design of Allpass Filters Using Least Squares Phase Error Criterion," *IEEE Trans. CAS II*, Vol.41,No.1,pp.40-48,1994.

[6] T.Q.Nguyen,T.I.Laakso and R.D.Koilpillai: "Eigenfilter Approach for the Design of Allpass Filters Approximating a Given Phase Response," *IEEE Trans.SP*,Vol.**SP-42**,pp.2257-2263,1994.

[7] M.Ikehara,M.funaishi and H.Kuroda: "Design of Complex All-Pass Networks Using Remez Algorithm,"*IEEE Trans. Circuits and Systems II*,No.8,pp.549-556,1992

[8] B.Yegnanarayana: "Design of Recursive Group-Delay Filters by Autoregressive Modeling,"*IEEE Trans. ASSP*,Vol.**ASSP-30**,No.4,pp.632-637,1982.

[9] H.W.Schuessler and P.Steffen: "On the Design of All-passes with Prescribed Group-Delay,"*IEEE ISCAS*,pp.1313-1316,1990.