Solution to the Orthogonal M-Channel Bandlimited Wavelet Construction Proposition*

B. Tennant, R. M. Rao

Harris RF Communications, Rochester Institute of Technology

ABSTRACT

While bandlimited wavelets and associated IIR filters have shown serious potential in areas of pattern recognition and communications, the dyadic Meyer wavelet is the only known approach to construct bandlimited orthogonal decomposition. The sinc scaling function and wavelet are a special case of the Meyer. Previous works have proposed a M-Band extension of the Meyer wavelet without solving the problem. One key contribution of this paper is the derivation of the correct bandlimits for the scaling function and wavelets to guarantee an orthogonal basis. In addition, the actual construction of the wavelets based upon these bandlimits is developed. A composite wavelet will be derived based on the M scale relationships from which we will extract the wavelet functions. The proper solution to this task is proposed which will generate associated filters with the knowledge of the scaling function and the constraints for M-band orthogonality.

1. INTRODUCTION

Generally in applications of signal processing [3] compactly supported wavelets can be used in analyzing any given signal. However, bandlimited orthogonal wavelets have recently shown potential in areas of pattern recognition and communications [1,2]. We address the problem of designing a M-Band perfect reconstruction (PR) filter bank based on a bandlimited function. Currently the dyadic case of the Meyer construction is well understood, yet a M-Band extension of the theory would prove to be useful in the above mentioned applications. The problem of M-Band Meyer wavelets was first touched upon in the work of Jones [2] in the context of multi carrier modulation. Jones developed a wavelet set for use with orthogonally multiplexed communication channels. It will be shown here that the conclusions developed in [2] for the scaling function are not sufficient to guarantee proper wavelet generation.

The goal of this paper is to develop the theory of the M-Band bandlimited wavelet transform and use it to generate a new class of wavelets based on the Meyer wavelet and scaling function. It will be shown that the M-Band solution is a non-trivial extension of the dyadic Meyer wavelet construction. Section II will be dedicated to developing the theory of M-Band bandlimited wavelet transforms. We will generalize the dyadic Meyer formulation such that is leads to the development of the scaling function, wavelets and associated filters. Section III will present an example of the scaling and wavelet functions for M=3.

* Work partially made possible by a grant from the Gleason Foundation.

2. THE M-BAND BANDLIMITED WAVELET

2.1 Magnitude Solution

For the two band Meyer wavelet case there is a clearly defined scaling function and wavelet. When we apply this theory to the M - Band wavelet we lose the clean distinction between individual wavelets and the scaling function. Instead we can determine a composite wavelet in which are hidden the M-1 wavelets that we are interested in. To determine this composite wavelet we must first determine a form for the scaling function in the frequency domain for the M-Band case. With this determined a series of strict rules can be applied to extract all wavelets in a single iteration. The well known Poisson Summation formula clearly deals only with the magnitude of the filters response, implying that the phase information will need to be determined as a separate step in the filter bank analysis. We start with the filter equation given in (1).

$$\Phi(M\omega) = H(\omega)\Phi(\omega) \quad (1)$$

For the bandlimited case, $\Phi(\omega)$ is bandlimited to a frequency ω_m, i.e. $\Phi(\omega)=0$ for $|\omega|>\omega_m$. Then, from (1), $H(\omega)=0$ for the region between ω_m/M and π . Therefore,

$$\omega_m \le M\pi \quad (2)$$

The Poisson Summation formula clearly indicates that the scaling function must have a bandwidth that is greater than π. We therefore rewrite the ω_m as

$$\omega_m = \pi + \alpha \quad (3)$$

A restriction on α is that it must be a value that is greater than zero. From (1) we can also determine that

$$\begin{aligned} |\Phi(\omega)|^2 &= 1 \quad \text{for} \quad 0 \le \omega \le \pi - \alpha \\ |\Phi(\omega)|^2 &= 0 \quad \text{for } \omega \ge \pi + \alpha \end{aligned} \quad (4)$$

We turn now to the filter that is defined by the scaling function in (1). The filter, $H(\omega)$, is discrete which forces the frequency response to be 2π periodic. For our purposes we will write this as

$$H(\omega) = H(\omega + 2\pi) = H^*(2\pi - \omega) \quad (5)$$

This, with the Poisson Summation formula, allows us to define regions of support for the filter itself. From(2),(3) and (5),

$$\begin{aligned} H(\omega) &\ne 0 \quad \text{for } |\omega| \le \frac{\pi+\alpha}{M} \\ H(\omega) &= 0 \quad \text{for } \frac{\pi+\alpha}{M} < \omega < \left[2\pi - \left(\frac{\pi+\alpha}{M}\right)\right] \end{aligned} \quad (6)$$

The periodic nature of $H(\omega)$ implies that we can write at the boundary of (6)

$$H\left(2\pi - \frac{\pi+\alpha}{M}\right) = H\left(\frac{(2M-1)\pi - \alpha}{M}\right) = 0 \quad (7)$$

The boundary condition is found when the stop band of the H(ω) filter lines up directly with the start of the transition band in its shifted version. This provides us with the limit that is necessary for the development of our α term. From this we can determine the initial range on our α condition given the scaling function as

$$0 \le \alpha \le \pi\left(\frac{M-1}{M+1}\right) \tag{8}$$

In order to guarantee a valid scaling function the transition band must satisfy the Poisson Summation formula. We write this as

$$|\Phi(\omega)| = \begin{cases} 1 & 0 \le \omega < \pi - \alpha \\ \gamma(\omega) & \pi - \alpha \le \omega < \pi + \alpha \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

The function γ(ω) satisfies the transition band defined by the Poisson Summation formula

$$\gamma(\omega) \Rightarrow |\Phi(\omega)|^2 + |\Phi(\omega + 2\pi)|^2 = 1 \tag{10}$$

If we let M=2 in (8) we see that it reduces to the correct value for the two band Meyer wavelet case for which the scaling function is bandlimited to 4π/3. Although (8) was also developed by W.W. Jones [2] we will see in following sections that this boundary on α is not sufficient to guarantee a valid M – Band wavelet system. We will actually need to place further restrictions on α to properly satisfy orthogonality.

2.2 The Composite Wavelet

In order to determine a set of M-1 valid wavelet functions $\psi_0(t)$, $\psi_1(t)$,…,$\psi_{M-2}(t)$, we introduce the concept of a composite wavelet θ(t) whose Fourier transform magnitude satisfies

$$|\Theta(\omega)|^2 = \sum_{m=0}^{M-2} |\Psi_m(\omega)|^2 \tag{11}$$

From general discussions on orthogonal M-Band wavelet filters, we can establish that the M-Band relationship between the composite wavelet and the scaling function is given as

$$|\Phi(\omega)|^2 = |\Phi(M\omega)|^2 + |\Theta(M\omega)|^2 \tag{12}$$

From equation **(12)**, the composite wavelets shape can be determined once we have developed an expression for the associated scaling function. We have developed the rules for frequency domain relationships between the individual wavelets and the composite wavelet provided in (11) from which the form of each individual bandlimited wavelet can then be found. The 3db points for the composite wavelet in the 2 band case occur at both π and 2π. If we assume that the 3db points in the M-Band case occur at π and Mπ (see Figure 1) we can write the average bandwidth of each wavelet as

$$\bar{B}_\Psi = \frac{M\pi - \pi}{(M-1)} = \pi \tag{13}$$

We have already illustrated that the scaling function essentially consists of three regions, a pass band, a transition band and a stop band. The pass band is identically equal to one in the range $0 \le \omega < (\pi - \alpha)$. Similarly the stop band is zero for the region where $\omega > \pi + \alpha$. The transition band has also been defined to exist such that (10) is completely satisfied.

$$\Phi(\omega) = \gamma(\omega) \text{ in the region } (\pi - \alpha) \le \omega < (\pi + \alpha) \tag{14}$$

The function γ(ω) must be such that

$$\gamma(\omega)^2 + \gamma(2\pi - \omega)^2 = 1$$

This equation is only important in the regions defined in (9). We can define a function $\tilde{\gamma}(\omega)$ that is the mirror image of γ(ω) in this region so that the above can be rewritten as

$$\gamma(\omega)^2 + \tilde{\gamma}(\omega)^2 = 1 \tag{15}$$

The composite wavelet is a band pass function that has five bands that of are interest to us, each of which can be determined from the shape of the scaling function. In Figure 1 we can see that the transition from stop band to start band is defined as $\tilde{\gamma}(\omega)$. The composite wavelet maintains the shape of a dilated scaling function after this transition. If we expand this function in **(12)** so that it is a function of ω not Mω then the shape of the function is defined not by Φ(ω) but by $\Phi\left(\frac{\omega}{M}\right)$. This dilation causes the transition from pass band to stop band to be governed by the function $\gamma\left(\frac{\omega}{M}\right)$. We have already determined a function for γ(ω) in previous section when we defined the scaling function, we therefore have a complete mathematical model for the magnitude of the composite wavelet

Assume that the transition band of each wavelet embedded within the composite wavelet occurs such that the center of the transition is at multiples of the average wavelet bandwidth, π. In order to maintain the shape of the composite wavelet and its symmetry, we must have for each wavelet

$$|\Psi_m(\omega)|^2 = \begin{cases} \left|\tilde{\gamma}\left(\frac{\omega}{m+1}\right)\right|^2 & \textit{for } (m+1)(\pi-\alpha) \le \omega < (m+1)(\pi+\alpha) \\ 1 & \textit{for } (m+1)(\pi+\alpha) \le \omega < (m+2)(\pi+\alpha) \\ \left|\gamma\left(\frac{\omega}{m+2}\right)\right|^2 & \textit{for } (m+2)(\pi-\alpha) \le \omega < (m+2)(\pi+\alpha) \\ 0 & \textit{otherwise} \end{cases} \tag{16}$$

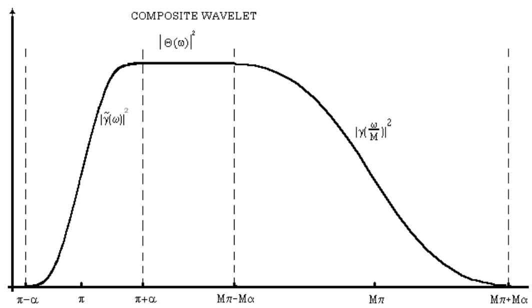

Figure 1 - The composite wavelet and its band edges

The design is such that for any overlapping transition band between two neighboring wavelets the following holds true

$$\gamma\left(\frac{\omega}{m}\right)^2 + \tilde{\gamma}\left(\frac{\omega}{m}\right)^2 = 1 \tag{17}$$

Previously, we defined α such that the scaling function and the composite wavelet correctly interact. There is an issue with this value that arises when dealing with individual wavelets. If we choose α to be exactly $\pi\frac{M-1}{M+1}$ then the start of the transition band of the last wavelet will extend past the start of the transition band of the first wavelet. This will cause all wavelet Fourier transforms to overlap, thus violating the Poisson summation formula. We must require that only neighboring wavelet bands overlap in their transition bands. For this to hold true we need to examine the bounds on the last transition to pass band, (M-1), and the second to last transition to stop band (M-2). In the best case scenario these two bands can just touch one another, we therefore write

$$(M-1)\pi-(M-1)\alpha \ge (M-2)\pi+(M-2)\alpha \tag{18}$$

Solving this equation for α, uniquely determines a new value of α that limits our choice for the original scaling function design.

$$\alpha \le \frac{\pi}{2M-1} \tag{19}$$

The result in (19) provides the correct value for the wavelet design and is tighter for M>2 than the one presented by Jones in (8). Generating a scaling function, as we did in (9), using the α term given in (19) generates the correct bandlimited orthogonal M–Band wavelets. It should be noted that the above equation reduces to the basic Meyer scaling function and boundary conditions when M=2. In addition, at M=2 Jones' results coincidently match those in (19). Also note as M increases, α approaches zero, moving our scaling function closer to the ideal Shannon scaling function. This shows that as M increases our filters in the system move toward the uniform bandwidth defined in (13).

2.3. Phase Solution

Due to the non ideal nature of our wavelet filters, there are overlapping regions in the magnitude solution. Phase solutions for these functions that are relative to the scaling filter, H(ω), can provide for the proper cancellation needed for our system to satisfy the orthogonality criterion. From the orthogonality of the scaling function and wavelet we can write

$$\sum_{k=-\infty}^{\infty}\Phi(\omega+2\pi k)\Psi_0^*(\omega+2\pi k)=0 \tag{20}$$

We are also aware of the M scale relationship between of the frequency response of the scaling function and wavelet as

$$\Phi(\omega)=H\left(\frac{\omega}{M}\right)\Phi\left(\frac{\omega}{M}\right)\ and\ \Psi_0(\omega)=G_0\left(\frac{\omega}{M}\right)\Phi\left(\frac{\omega}{M}\right) \tag{21}$$

If we substitute (21) back into (20) and reduce the expression with some basic knowledge wavelet filters we can arrive at the solution

$$\sum_{k=-\infty}^{\infty}H\left(\omega+\frac{2\pi k}{M}\right)G_0^*\left(\omega+\frac{2\pi k}{M}\right)=0 \tag{22}$$

The bounds on (22) become critical in the phase solution development and it should be clearly noted that they extend across all integer k. Similarly we develop an expression for all adjacent wavelets as

$$\sum_{k=-\infty}^{\infty}G_{m-1}\left(\omega+\frac{2\pi k}{M}\right)G_m^*\left(\omega+\frac{2\pi k}{M}\right)=0\quad m\in\{1,2\cdots M-2\} \tag{23}$$

We are only interested in the phase for the adjacent wavelets and scaling function due to the physical structure of the system. The filters are designed such that only neighboring functions overlap. Non-adjacent wavelets immediately satisfy the orthogonal relationships due to the product always being identically zero, which is attributed to these filters non overlapping regions of support. We define a new variable, **q**, that indexes the overlapping regions as follows

$$q=\{0,1,2\ldots M-2\}$$

With q, we develop an expression for the required shift to properly satisfy cancellation of these overlapping regions. If we define a function z(q) as

$$z(q)=\begin{cases}\dfrac{M}{2(q+1)} & for\ \ 0\le q<\dfrac{M}{2}\\[2mm] \dfrac{M}{2M-2(q+1)} & for\ \ \dfrac{M}{2}\le q\le M-2\end{cases} \tag{24}$$

The resulting phase shift required to satisfy the orthogonality criterion becomes

$$\rho_q(\omega)=-e^{-jz(q)\omega} \tag{25}$$

This results in the relationship between scaling function and wavelets as

$$\begin{aligned}G_0(\omega)&=\rho_0(\omega)H^*\left(\omega+\frac{2\pi}{M}\right)\\ G_m(\omega)&=\rho_m(\omega)G_{m-1}^*\left(\omega+\frac{2\pi}{M}\right)for\ m\in\{1,2\ldots M-2\}\end{aligned} \tag{26}$$

Notice that the z(q) term is designed to rotate the shifts around the unit circle at specific points so that cancellation properly occurs, in general resulting non-integer shifts. Also each shift is with respect to the previous wavelet or scaling function. This implies that the total shift is accumulative, and with respect to the scaling function (which is defined with a phase of zero) the relative shift continues to increase.

3. EXAMPLE

We will examine a Meyer wavelet construction for the case where M=3. For illustrative purposes we will assume the worst case α, in practical situations the filters can be made better by choosing a smaller value for this parameter. We start with the knowledge that our system is broken into three separate bands. Referring to (19) we see that

$$\alpha=\frac{\pi}{(2\cdot 3-1)}=\frac{\pi}{5}$$

A γ(ω) function for the scaling function and associated wavelets can be constructed utilizing this parameter. We have shown that any function that satisfies (10) will generate an acceptable M-Band system. For this case we have chosen γ(ω) to take on a section of the cosine function.

From γ(ω) and our knowledge of the bounds on the scaling function, we can now readily sketch out the remainder of Φ(ω). Notice that the transition band is the heavy line in Figure 2.

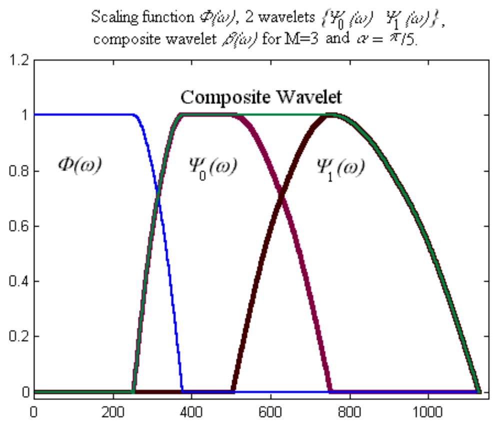

Figure 2– Depicts the Fourier transform squared magnitude of the scaling function and wavelets embedded within the Fourier transform squared magnitude composite wavelet for M=3.

The M scale relationship that we developed in (12) allows to quickly develop the composite wavelet that will contain all embedded wavelets. The rising edge of the composite wavelet is simply $\tilde{\gamma}(\omega)$. We have also established that the shape of the composite wavelet is derived from the shape of the dilated scaling function. This implies that the falling edge of the filter is governed by the shape given by $\gamma\left(\frac{\omega}{3}\right)$. As previously mentioned there are two wavelets embedded in the composite wavelet design. These two functions are also uniquely described by the $\gamma(\omega)$ that we arrived at. The midpoint of each transition occurs at $(m+1)\pi$ where m is the wavelet function index $\{0,1\}$. Each successive wavelet dilates the $\gamma(\omega)$ function by another factor of M. The passband region of each wavelet is identically one and the stop band identically zero, from this we can arrive at both embedded wavelets. The scaling function, composite wavelet and the two Meyer wavelets are all clearly illustrated in Figure 3.

The scaling function and the corresponding wavelets are also smooth and closely resemble the two band Meyer wavelet and scaling function. One point that we should notice in this example is the duration under which the signal energy is spread. We will find that as we increase our α term the energy will be supported less compactly than in the case above. The corresponding filters can be found utilizing (21) and (26) and clearly result in bandlimited filters which closely resemble their parent scaling function and wavelets.

Figure 4 provides an additional example of a M=10 scenario. Here we have chosen α to again be the lower limit of the bound. It can be seen from figure 4 that the frequency responses transition bands have greater overlap for larger values of M. This is a direct result of (16). These bands can be made tighter by reducing the value of α which will result in the time domain filters energy being spread across a longer duration. The composite wavelet in figure 4 is the dark band under which the nine wavelet frequency responses are contained.

4. Conclusion

For the first time wavelets have been produced for orthogonal M-Band bandlimited decomposition, through a non-trivial extension of the 2-band Meyer wavelet case. It has been shown that the M–Band Meyer wavelet system generates a PRFB set if we impose the proper constrains on the scaling function. Previous developments of this M-Band extension were shown to be insufficient in their derivation of boundary conditions, resulting in functions that did not properly satisfy the wavelet criteria.

One advantage that our design offers is in the smoothness of the Meyer wavelet. This is particularly useful in image processing systems where the transitions tend to be hard edges. In areas where the 2 Band bandlimited filters are used [5], we now have a M-Band extension. Another clear advantage to this development is the lack of restriction that we place on M, which can be either even or odd valued. It is believed that this will further enhance the analysis properties of the wavelet under these such conditions.

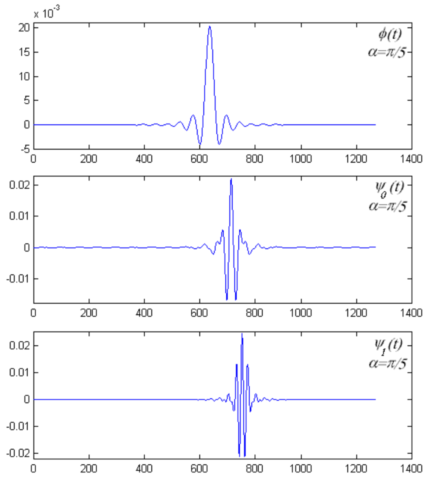

Figure 3 - Time domain representation of the Meyer wavelets and scaling function for M=3.

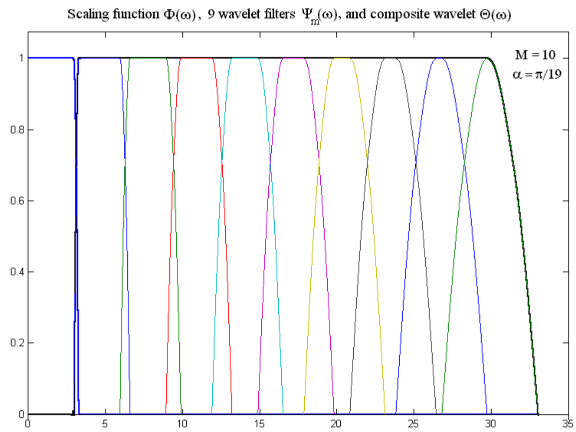

Figure 4 – The Fourier transform squared magnitude of the scaling function, composite wavelet and the 9 corresponding embedded wavelet functions in a M=10 system.

REFERENCES

1. J.O. Chapa and R.M. Rao, "Algorithms for Designing Wavelets to Match a Specified Signal," IEEE Trans. Signal Proc., Dec. 2000
2. W. W. Jones "A Unified Approach to Orthogonally Multiplexed Communications Using Wavelet Basis and Digital Filter Banks", Ph.D. Dissertation, Univ. of Ohio, Aug. 1994
3. M. Vetterli and C. Herley. "Wavelets and Filter Banks: Theory and Design" IEEE, Trans on ASSP, pages 2207-2232, Sept 1992
4. A.S. Bopardikar, R.M. Rao and B.S. Adiga, "PRCC Filter Banks, Matched Sampling Systems and Implementation of Bandlimited Discrete Wavelet Transform," IEEE Transactions on Signal Processing, August 2000.
5. G.G. Walter and J. Zhang, "Orthonormal Wavelets with Simple Closed-Form Expressions," IEEE Trans. on Signal Proc. Aug. 1998
6. P. P. Vaidyanathan, Multirate Systems and Filter Banks, Prentice Hall, 1992