BASES FOR THE WHOLE SAMPLE SYMMETRIC FILTER BANKS USING A COMPLEX ALLPASS FILTER

Magdy Tawfik Hanna
Department of Engineering Mathematics and Physics
Cairo University / Fayoum Branch
Fayoum, Egypt

ABSTRACT

This work is concerned with the design problem of a whole sample symmetric filter bank where the lowpass and highpass filters are defined in terms of a complex allpass filter. The design problem reduces to the evaluation of the coefficients of this allpass filter. The regularity requirement of the filter bank leads to a set of flatness constraints. The required frequency selectivity of the filters is expressed as another set of constraints. The combined set of constraints leads to a generalized eigenvalue problem that can be reduced to a regular eigenvalue problem by conjecturing a certain matrix to be nonsingular. The eigenvector corresponding to the largest positive eigenvalue of the last problem is the sought vector of the coefficients of the allpass filter.

I. INTRODUCTION

The importance of the symmetry of the wavelet function stems from the linear phase of the filters of the filter bank corresponding to this wavelet. Although the FIR filter enjoys a linear phase, the only symmetric wavelet corresponding to the FIR filter is the Haar wavelet which is discontinuous [1]. In order to obtain real-valued orthonormal wavelet bases with better regularity than the Haar wavelet, Herley and Vetterli proposed a class of IIR filters where the filter bank can be constructed using real allpass filters [2]. Zhang et. al. proposed a class of real-valued orthonormal symmetric wavelet bases where the associated whole sample symmetric (WSS) paraunitary filter banks are composed of a single complex allpass filter [3]. They first arrived at the form of the transfer function of the allpass filter, which satisfies the symmetry and orthonormality conditions of wavelets. They second attacked the design problem of paraunitary filter banks given the fact that the goals of wavelet regularity and filter's frequency selectivity are in conflict. They looked for filters having a given degree of flatness and optimal phase response of the complex allpass filter.

The main objective of the present paper is to rectify the work done in [3] and extend it. First a matrix that has been erroneously ignored in the eigenvalue formulation of the problem in [3] will be brought out. Second another matrix will be conjectured to be nonsingular and consequently the generalized eigenvalue problem will be reduced to an ordinary eigenvalue problem to be solved for the eigenvector corresponding to the largest positive eigenvalue. Third some error that crept into [3] will be corrected.

II. THE COMPLEX ALLPASS FILTER

Let H(z) and G(z) be respectively the lowpass and highpass filters of the two-band paraunitary filter bank that generates orthonormal wavelet basis. In order to construct a WSS filter bank where both the wavelet and scaling functions are symmetric, the numerator degrees of H(z) and G(z) should be even [2]. H(z) and G(z) can be constructed from a single complex allpass filter A(z) as follows [4]:

$$H(z) = \frac{1}{2}\left[A(z) + \tilde{A}(z)\right] \tag{1}$$

$$G(z) = \frac{z^{-1}}{j2}\left[A(z) - \tilde{A}(z)\right] \tag{2}$$

where $\tilde{A}(z)$ is obtained from $A(z)$ by complex conjugating its coefficients. The linear phase and orthonormality conditions necessitate that $A(z)$ has the form [3]:

$$A(z) = e^{j\eta} z^{-N} \frac{a_0 + ja_1 z + a_2 z^2 + ja_3 z^3 + \cdots}{a_0 - ja_1 z^{-1} + a_2 z^{-2} - ja_3 z^{-3} + \cdots}$$
$$\frac{\cdots + ja_3 z^{N-3} + a_2 z^{N-2} + ja_1 z^{N-1} + a_0 z^N}{\cdots - ja_3 z^{-(N-3)} + a_2 z^{-(N-2)} - ja_1 z^{-(N-1)} + a_0 z^{-N}} \tag{3}$$

where a_n's are real coefficients, $a_0 = 1$, N is an even integer, and $\eta = \pm 0.25\pi$ or $\eta = \pm 0.75\pi$. The phase response $\theta(\omega)$ of $A(z)$ is given by:

$$\theta(\omega) = \eta + 2\phi(\omega) \tag{4}$$

where

$$\phi(\omega) = \tan^{-1} \frac{N(\omega)}{D(\omega)} . \tag{5}$$

Since N is an even integer, it can be expressed as N = 2M and one gets:

$$N(\omega) = \begin{cases} \sum_{n=0}^{0.5M-1} a_{2n+1} \cos(M - 2n - 1)\omega & \text{if M is even} \\ 0.5a_M + \sum_{n=0}^{0.5(M-3)} a_{2n+1} \cos(M - 2n - 1)\omega & \text{if M is odd} \end{cases} \tag{6}$$

and

$$D(\omega)=\begin{cases}0.5a_M+\sum_{n=0}^{0.5M-1} a_{2n}\cos(M-2n)\omega & \text{if M is even}\\ \sum_{n=0}^{0.5(M-1)} a_{2n}\cos(M-2n)\omega & \text{if M is odd}\end{cases} \tag{7}$$

III. THE FLATNESS CONDITIONS

.

The regularity of the filters can be expressed as the following flatness conditions:

$$\left.\frac{\partial^k\left|H\left(e^{j\omega}\right)\right|}{\partial\omega^k}\right|_{\omega=\pi}=0 \quad ,k=0,\cdots,K-1 \tag{8}$$

$$\left.\frac{\partial^k\left|G\left(e^{j\omega}\right)\right|}{\partial\omega^k}\right|_{\omega=0}=0 \quad ,k=0,\cdots,K-1\ . \tag{9}$$

The above two sets of conditions are equivalent since H(z) and G(z) are orthogonal. Zhang et. al. [3] showed that the K conditions of (9) can be reduced to the following $0.5K$ explicit conditions:

$$VD\,a=0 \tag{10}$$

where V is the $0.5K\times(M+1)$ Vandermonde matrix:

$$V=\begin{bmatrix}1 & 1 & \cdots & 1 & 1\\ M^2 & (M-1)^2 & \cdots & 1 & 0\\ M^4 & (M-1)^4 & \cdots & 1 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ M^{K-2} & (M-1)^{K-2} & \cdots & 1 & 0\end{bmatrix} \tag{11}$$

and D is the diagonal matrix:

$$D=Diag\{d_0,d_1,\cdots,d_M\} \tag{12}$$

with its elements given by:

$$d_i=\begin{cases}1 & \text{i even}\\ \cot 0.5\eta & \text{i odd}\end{cases} \quad ,i=0,1,\cdots,M-1 \tag{13}$$

$$d_M=\begin{cases}0.5 & \text{M even}\\ 0.5\cot 0.5\eta & \text{M odd}\end{cases} \tag{14}$$

The elements of the column vector a in (10) are the coefficients of the allpass filter $A(z)$ of (3), i.e.,

$$a=\begin{bmatrix}a_0 & a_1 & \cdots & a_M\end{bmatrix}^T . \tag{15}$$

The number of conditions $0.5K$ in (10) lies in the range $0\le 0.5K\le M$ where the extreme case of $K=2M$ corresponds to the maximally flat filters.

IV. FREQUENCY SELECTIVITY CONDITIONS

Since the allpass filter is completely specified by its phase angle $\theta(\omega)$, the design problem reduces to the evaluation of $\theta(\omega)$. It can be shown that the desired phase response of $A(z)$ is given by [3]:

$$\theta_d(\omega)=\begin{cases}0 & 0\le\omega\le\omega_p\\ \pm 0.5\pi & \omega_s\le\omega\le\pi\end{cases} \tag{16}$$

where ω_p and ω_s are the cutoff frequencies of the passband and stopband of H(z) respectively and $\omega_p+\omega_s=\pi$. The corresponding desired $\phi(\omega)$ can be obtained from (4) as:

$$\phi_d(\omega)=\begin{cases}-0.5\eta & 0\le\omega\le\omega_p\\ \pm 0.25\pi-0.5\eta & \omega_s\le\omega\le\pi\end{cases} \tag{17}$$

It can be shown from (5)-(7) that

$$\phi(0.5\pi)=\begin{cases}0 & \text{if M is even}\\ \pm 0.5\pi & \text{if M is odd}\end{cases} \tag{18}$$

and

$$\phi(\omega)=\begin{cases}-\phi(\pi-\omega) & \text{if M is even}\\ \pm\pi-\phi(\pi-\omega) & \text{if M is odd}\end{cases} \tag{19}$$

In order for $\phi_d(\omega)$ of (17) to produce the symmetry expressed by (19), the angle η should be:

$$\eta=\begin{cases}\pm 0.25\pi & \text{if M is even}\\ \pm 0.75\pi & \text{if M is odd}\end{cases} \tag{20}$$

Because of the symmetry of $\phi(\omega)$ expressed by (19) about the frequency $\omega=0.5\pi$, the design problem reduces to the approximation of $\phi_d(\omega)$ in the passband $\omega\in[0,\omega_p]$.

In order to get an equiripple phase response $\phi(\omega)$, one imposes the following conditions:

$$\phi(\omega_i)-\phi_d(\omega_i)=(-1)^{i+l}\delta_p \quad ,i=0,1,\cdots,M-0.5K \tag{21}$$

where the ω_i's are extremal frequencies in the passband $\omega\in[0,\omega_p]$ arranged as:

$$\omega_p=\omega_0>\omega_1>\cdots>\omega_{M-0.5K}\ge 0\ . \tag{22}$$

In (21), δ_p is the phase error to be minimized and the integer l is selected such that δ_p will always be positive. Some thought about $\phi_d(\omega)$ of (17) and an optimal equiripple $\phi(\omega)$ superimposed on it will lead to the values of l given in Table 1. One should mention that those values of l have been erroneously interchanged in [3].

From (21), (5) and (17), one gets:

$$\tan\left[\phi(\omega_i)-\phi_d(\omega_i)\right]=\frac{D(\omega_i)+N(\omega_i)\cot 0.5\eta}{D(\omega_i)\cot 0.5\eta-N(\omega_i)} = (-1)^{i+l}\delta \tag{23}$$

where $\delta=\tan\delta_p$.

First one considers the case of even M and substitutes the expressions of $N(\omega)$ and $D(\omega)$ from (6) and (7) in (23) to get:

$$\begin{aligned}&\left[\cos M\omega_i \quad \cos(M-1)\omega_i \quad \cdots \quad \cos\omega_i \quad 1\right]\\&\left[a_0 \quad a_1 \cot 0.5\eta \quad \cdots \quad a_{M-1}\cot 0.5\eta \quad 0.5a_M\right]^T\\&=(-1)^{i+l}\,\delta\left[\cos M\omega_i \quad \cos(M-1)\omega_i \quad \cdots \quad \cos\omega_i \quad 1\right]\\&\left[a_0 \cot 0.5\eta \quad -a_1 \quad \cdots \quad -a_{M-1} \quad 0.5a_M \cot 0.5\eta\right]^T\end{aligned} \qquad (24)$$

.

In order to put the above equation in a compact form, one defines the diagonal matrix T as:

$$T = Diag\{t_0, t_1, \cdots, t_M\} \qquad (25)$$

where the diagonal elements are given by:

$$t_i = \begin{cases}(-1)^l \cot 0.5\eta & \text{i even}\\(-1)^{l+1} & \text{i odd}\end{cases}, i = 0,1,\cdots, M-1 \qquad (26)$$

$$t_M = \begin{cases}0.5(-1)^l \cot 0.5\eta & \text{M even}\\0.5(-1)^{l+1} & \text{M odd}\end{cases} \qquad (27)$$

Using the above definition and its counterpart for matrix D given by (12)-(14), one can express (24) as:

$$\begin{aligned}&\left[\cos M\omega_i \quad \cos(M-1)\omega_i \quad \cdots \quad \cos\omega_i \quad 1\right] D\, a =\\&(-1)^i\,\delta\left[\cos M\omega_i \quad \cos(M-1)\omega_i \quad \cdots \quad \cos\omega_i \quad 1\right] T\, a\end{aligned} \qquad (28)$$

where vector a is defined by (15).

Second one considers the case of odd M and substitute (6) and (7) in (23) to get:

$$\begin{aligned}&\left[\cos M\omega_i \quad \cos(M-1)\omega_i \quad \cdots \quad \cos\omega_i \quad 1\right]\\&\left[a_0 \quad a_1 \cot 0.5\eta \quad \cdots \quad a_{M-1} \quad 0.5a_M \cot 0.5\eta\right]^T\\&=(-1)^{i+l}\,\delta\left[\cos M\omega_i \quad \cos(M-1)\omega_i \quad \cdots \quad \cos\omega_i \quad 1\right]\\&\left[a_0 \cot 0.5\eta \quad -a_1 \quad \cdots \quad a_{M-1}\cot 0.5\eta \quad -0.5a_M\right]^T\end{aligned} \qquad (29)$$

Using the definition of the diagonal matrices D and T, one verifies that the above equation can be compactly expressed as (28). Therefore (28) holds irrespective of M being even or odd.

Writing (28) for the $(M-0.5K+1)$ extremal frequencies of (22), one gets:

$$C\,D\,a = \delta\; S\,C\,T\,a \qquad (30)$$

where C is the $(M-0.5K+1)\times(M+1)$ matrix:

$$C = \begin{bmatrix}\cos M\omega_0 & \cos(M-1)\omega_0 & \cdots & 1\\ \cos M\omega_1 & \cos(M-1)\omega_1 & \cdots & 1\\ \vdots & \vdots & \vdots & \vdots\\ \cos M\omega_{M-0.5K} & \cos(M-1)\omega_{M-0.5K} & \cdots & 1\end{bmatrix} \qquad (31)$$

and S is a diagonal matrix of order $(M-0.5K+1)$ defined by:

$$S = Diag\{1,-1,1,-1,\cdots,(-1)^{M-0.5K}\}. \qquad (32)$$

One should point out that matrix S did not appear in the treatment of [3] since it was erroneously ignored.

V. COMBINED FLATNESS AND FREQUENCY SELECTIVITY

In order to achieve the goals of wavelet regularity and filter's frequency selectivity, one should aspire to achieve both the flatness conditions (10), the equiripple conditions (30) and to minimize the error δ. Combining the two sets of constraints (10) and (30) one gets:

$$P\,a = \delta\,Q\,a \qquad (33)$$

where

$$P = \begin{pmatrix}V\\C\end{pmatrix} D \quad \text{and} \quad Q = \begin{pmatrix}O\\S\,C\end{pmatrix} T\,. \qquad (34)$$

In Appendix B, one will conjecture that matrix P is nonsingular and actually an extensive numerical investigation showed that this is true. Consequently the generalized eigenvalue problem of (33) reduces to the ordinary eigenvalue problem:

$$A\,a = \lambda\,a \qquad (35)$$

where

$$A = P^{-1}Q \quad \text{and} \quad \lambda = 1/\delta\,. \qquad (36)$$

Since δ should be positive and minimum, one should seek the largest positive eigenvalue of A since (36) implies that $\delta_{min} = 1/\lambda_{max}$. Consequently the vector of coefficients a which satisfies the combined constraint (35) and minimizes the error δ is the eigenvector of matrix A corresponding to the maximum positive eigenvalue.

Since the extremal frequencies ω_i's in (21) are not known beforehand, one should use the Remez exchange algorithm [3].

In the extreme case of $K = 0$, one has the minimax frequency response and matrix A of (36) reduces to:

$$A = D^{-1}C^{-1}S\,C\,T \qquad (37)$$

as can be seen from (34). The fact that matrix C in this case is nonsingular will be proved in appendix A.

VI. CONCLUSION

A complex allpass filter has been designed and used as the building block for the lowpass and highpass filters of a whole sample symmetric filter bank. The problem formulation has involved a generalized eigenvalue problem that has been reduced to a regular eigenvalue problem by showing that a certain matrix is nonsingular.

APPENDIX A

Fact: The $(M-0.5K+1)\times(M+1)$ matrix C defined by (31) has a full row rank.

Proof:

One starts by stating the following definition and theorems:

Definition: A set of $(M+1)$ functions $\{\phi_0(\omega), \phi_1(\omega), \cdots, \phi_M(\omega)\}$ on $[a,b]$ is a Chebyshev set on $[a,b]$ iff for all $\alpha_0, \alpha_1, \cdots, \alpha_M$ the linear combination $\sum_{i=0}^{M}\alpha_i\phi_i(\omega)$ has at most M distinct zeroes on $[a,b]$ [5].

Theorem 1: If $\{\phi_0(\omega),\cdots,\phi_M(\omega)\}$ is a Chebyshev set on $[a,b]$, then the square matrix:

$$\begin{pmatrix} \phi_0(\omega_0) & \cdots & \phi_M(\omega_0) \\ \phi_0(\omega_1) & \cdots & \phi_M(\omega_1) \\ \vdots & \vdots & \vdots \\ \phi_0(\omega_M) & \cdots & \phi_M(\omega_M) \end{pmatrix}$$

is nonsingular provided that $\omega_0,\cdots,\omega_M$ are distinct points in $[a,b]$ [5].

Theorem 2: $\{1,\cos\omega,\cos 2\omega,\cdots,\cos M\omega\}$ is a Chebyshev set on $[0,\pi]$ [6].

One first considers the case of $K=0$ and denotes the square matrix C of (31) by C_S. Since the frequencies $\omega_0,\cdots,\omega_M$ are distinct by virtue of (22) and lie in the interval $[0,\omega_p]\subset[0,\pi]$, the above two theorems imply that C_S is nonsingular.

In the case of $K>0$, matrix C of (31) will be formed by the first $(M-0.5K+1)$ rows of the nonsingular matrix C_S, and consequently it will have a full-row rank.

APPENDIX B

Conjecture: Matrix P defined by (34) is nonsingular and consequently matrix A defined by (36) has a rank of $(M-0.5K+1)$.

Steps:

1) Since the $0.5K\times(M+1)$ matrix V defined by (11) is a Vandermonde matrix and since $0.5K\le M$, it will have a full row rank, i.e.,

$$\rho(V)=0.5K\,. \qquad \text{(B1)}$$

2) The heuristic step: Since the two matrices V of (11) and C of (31) have full row ranks, the square partitioned matrix $\begin{pmatrix} V \\ C \end{pmatrix}$ is conjectured to have a full row rank and consequently to be nonsingular. Although this may not be true in general for any two matrices, the fact that V and C have completely different structures supports the conjecture.

3) Since the diagonal matrix D defined by (12)-(14) is nonsingular, one gets from (34) and the above step that:

$$\rho(P)=\rho\left(\begin{bmatrix} V \\ C \end{bmatrix}\right)=M+1\,. \qquad \text{(B2)}$$

4) Since matrix S defined by (32) is nonsingular, one gets:

$$\rho(S\,C)=\rho(C)=M-0.5K+1\,. \qquad \text{(B3)}$$

5) Since the diagonal matrix T defined by (25)-(27) is nonsingular, one gets from (34) and the above step that:

$$\rho(Q)=\rho\left(\begin{bmatrix} O \\ S\,C \end{bmatrix}\right)=\rho(S\,C)=M-0.5K+1\,. \qquad \text{(B4)}$$

6) From (36) and the above step, one gets:

$$\rho(A)=\rho(Q)=M-0.5K+1\,. \qquad \text{(B5)}$$

In the above development only the second step is heuristic. Actually extensive numerical investigation has shown that P is nonsingular.

REFERENCES

[1] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley, MA: Wellesley-Cambridge Press, 1996.

[2] C. Herley and M. Vetterli, "Wavelets and recursive filter banks," *IEEE Transactions on Signal Processing*, vol. 41, pp. 2536-2556, Aug 1993.

[3] X. Zhang, A. Kato and T. Yoshikawa, "A new class of orthonormal symmetric wavelet bases using a complex allpass filter," *IEEE Transactions on Signal Processing*, vol. 49, pp. 2640-2647, November 2001.

[4] P.P. Vaidyanathan, P.A. Regalia and S.K. Mitra, "Design of doubly complementary IIR digital filters using a single complex allpass filter with multirate applications," *IEEE Transactions on Circuits and Systems*, vol. CAS-34, pp. 378-389, April 1987.

[5] J.R. Rice, *The Approximation of Functions*, vol. 1, Reading, MA: Addison Wesley, 1964, pp. 55-66.

[6] J.H. McClellan and T.W. Parks, "Eigenvalue and eigenvector decomposition of the discrete Fourier transform," *IEEE Transactions on Audio and Electroacoustics*, vol. AU-20, pp. 66-74, March 1972.

Table 1: The values of the integer l in (21).

M	even		odd	
η	0.25π	-0.25π	0.75π	-0.75π
$\phi(0.5\pi)$	0	0	-0.5π	0.5π
l	0	1	1	0