# An Optimised Normalised LMF Algorithm For Sub-Gaussian Noise

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## Abstract

The least mean fourth (LMF) algorithm is known for its fast convergence and lower steady state error, especially under sub-Gaussian noise conditions. Meanwhile, the recent work on the normalised versions of LMF algorithm has further enhanced its stability and performance in both Gaussian and sub-Gaussian noise. For example, the normalised LMF (XE-NLMF) algorithm, recently developed, is normalised by the mixed signal power and error power, and weighted by a fixed mixed-power parameter. Unfortunately, this algorithm depends on the selection of this mixing parameter. To overcome this obstacle, in this work, a time-varying mixed-power parameter technique is introduced to optimise its selection. An enhancement in performance is obtained through the use of this procedure in both the convergence rate and steady-state error.

## 1 Introduction

The LMF algorithm belongs to the class of stochastic gradient descent based algorithms, similar to the least mean square (LMS) algorithm [1]. The power of LMF lies in its faster initial convergence and lower steady state error relative to the LMS algorithm. More importantly, its mean fourth error cost function is optimum for noise of sub-Gaussian nature [2], or light-tailed probability-density- function-like noise [3].

However, this higher order algorithm requires a much smaller step size to ensure stable adaptation [4]. Where, the error power three in the LMF gradient vector can cause devastating initial instability. Therefore, it causes unnecessary performance degradation. The solution that is proposed is to normalise the step size as developed in [5, 6].

Although, both normalisation techniques are quite similar, the XE-NLMF [6] offers more flexibility to gain, eventually more improvement in performance. The recursive equation for the XE-NLMF algorithm is defined as follows [6]:

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \frac{\gamma_{xe} e^3(n)\mathbf{x}(n)}{\delta + (1-\lambda)\|\mathbf{x}(n)\|^2 + \lambda\|\boldsymbol{e}(n)\|^2}, \quad (1)$$

where $\gamma_{xe}$ represents the step size, $\mathbf{w}(n)$ is the filter coefficient vector of the adaptive filter, $\mathbf{x}(n)$ is the input vector and $\boldsymbol{e}(n)$ is the error vector. It is shown in equation (1) that the LMF is normalised by the signal power and error power, which is balanced by a mixed-power parameter ($\lambda$). Combining signal power and error power has the advantage that the former normalise input signal, while the error power can dampen down the outlier estimation errors. Thus improving stability while still providing fast convergence speed.

In general the adaptation scheme defined in (1) can be set up into the following:

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \gamma_{xe} f(e(n))\mathbf{x}(n), \quad (2)$$

where $f(e(n))$ denotes a general scalar function of the output estimation error $e(n)$. Table 1 defines $f(e(n))$ for many famous special cases of (2).

This paper is an extension of the XE-NLMF algorithm [6]. Instead of a fixed value $\lambda$, a variable $\lambda(n)$ is proposed. The value of this mixed-power parameter will compromise between fast convergence and lower steady state error. Therefore, incorporating a variable value for the mixed-power parameter is prudent and desirable for better adaptive performance.

| Algorithm | $f(e(n))$ |
|---|---|
| LMS | $e(n)$ |
| LMF | $e^3(n)$ |
| NLMS | $\frac{e(n)}{\Vert\mathbf{x}(n)\Vert^2}$ |
| NLMF [5] | $\frac{e^3(n)}{\Vert\mathbf{x}(n)\Vert^2}$ |
| XE-NLMF [6] | $\frac{e^3(n)}{\delta+(1-\lambda)\Vert\mathbf{x}(n)\Vert^2+\lambda\Vert\boldsymbol{e}(n)\Vert^2}$ |
| Sign-LMS | $sign[e(n)]$ |

Table 1: Examples for $f(e(n))$.

This algorithm finds great applications in environments with highly dynamic channels. The time variations of the mixing parameter allow the algorithm to follow the changes in the channel as opposed to the same algorithm with fixed mixing parameter.

## 2. Variable Normalised XE-NLMF Algorithm

The mixing power parameter is confined to the interval [0,1] and will be weighted recursively to adjust the signal power, $\Vert\mathbf{x}(n)\Vert^2$, and error power, $\Vert\boldsymbol{e}(n)\Vert^2$, for the best performance. The error is defined as:

$$e(n) = d(n) - \mathbf{w}^{\mathbf{T}}(n)\mathbf{x}(n). \quad (3)$$

Here, we propose an error square feedback according to [7]:

$$\mu(n+1) = \nu\mu(n) + p(n)|e(n)e(n\text{-}1)|, \quad (4)$$

where the quantity $e(n)e(n\text{-}1)$ determines the distance of $\mathbf{w}(n)$ to the optimum weights, $|\cdot|$ denotes the absolute value operation, $p(n)$ is updated according to the sum of past three samples of $\lambda(n)$ in the following way:

$$p(n) = [\lambda(n\text{-}2) + \lambda(n\text{-}1) + \lambda(n)]a, \quad (5)$$

and $a$ and $\nu$ are constants.

With this averaging, the recursion curve of $\mu(n)$ can be more flexibly controlled. The error power estimate is then used to guide the $\lambda(n)$, as follows:

$$\lambda(n) = erf\{\mu(n)\}, \quad (6)$$

where $erf\{\cdot\}$ is an error function with the purpose to constrain the $\mu(n)$ to [0,1]. The parameters $\nu$ and $p$ are also restricted to the interval [0,1]. To avoid zero in the feedback loop, the initial value of $p$ is set at $p(0) = 0.5$.

This scheme provides an automatic adjustment to $\lambda(n)$ according to the estimation of error square. When the estimation error is large, $\lambda$ will approach unity and provide fast adaptation. While when the error is small (converged), $\lambda$ is adjusted to a smaller value for lower steady error. Based on this motivation, the proposed *variable normalised XE-NLMF algorithm* is expressed as follow:

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \frac{\gamma_{xe}e^3(n)\mathbf{x}(n)}{\delta+(1-\lambda(n))\Vert\mathbf{x}(n)\Vert^2+\lambda(n)\Vert e(n)\Vert^2}. \quad (7)$$

For this algorithm, the mixed-power parameter will take on the role of a forgetting factor.

## 3. Convergence Analysis

The mean convergence is studied with expectation of the weight error deviation, $\mathbf{v}(n) = \mathbf{w}(n) - \mathbf{w}^*$. Following the LMF convergence analysis in [2] and [6], the difference equation for the weight error is defined as follows:

$$E\{\mathbf{v}(n+1)\} = [\mathbf{I} - 3\gamma E\{\eta^2(n)\}\mathbf{R}]E\{\mathbf{v}(n)\}. \quad (8)$$

Hence the mean convergence for XE-NLMF is examined by replacing $\gamma$ with the normalised step size, as follows:

$$E\{\mathbf{v}(n+1)\} = [\mathbf{I} - \frac{3\gamma_{xe}E\{\eta^2(n)\}\mathbf{R}}{(1-\lambda(n))\sigma_x^2+\lambda(n)\sigma_e^2}]E\{\mathbf{v}(n)\}, \quad (9)$$

where $\Vert\mathbf{x}(n)\Vert^2 = \sigma_x^2$ and $\Vert\boldsymbol{e}(n)\Vert^2 = \sigma_e^2$. For $\mathrm{tr}[\mathbf{R}] = N\sigma_x^2$, a general condition for equation (9) to hold is:

$$\gamma_{xe} < \frac{1}{3N}\left(\frac{(1-\lambda(n))+}{\sigma_\eta^2}+\frac{\lambda(n)\sigma_e^2}{\sigma_\eta^2\sigma_x^2}\right). \quad (10)$$

Two conditions for stability are observed in equation (10). That is for $\lambda(n) = 1$ and $\lambda(n) = 0$. It therefore, can be seen as a balance normalisation in the variable normalised XE-NLMF algorithm. The effect of $\lambda$ can be seen in Figure 1.

The error is usually larger during the initial adaptation and gradually decreases toward a minimum. Therefore, the signal power, $\|\mathbf{x}(n)\|^2$, will act as a threshold to avoid large step size when the error converges to a minimum. The combination of $(1-\lambda(n))\|\mathbf{x}(n)\|^2 + \lambda(n)\|\boldsymbol{e}(n)\|^2$ has the advantage of normalising the input signal power and an improved stability where the $\|\boldsymbol{e}(n)\|^2$ will dampen down the outlier distribution of $e^3(n)$ in the recursive updating equation of XE-NLMF algorithm.

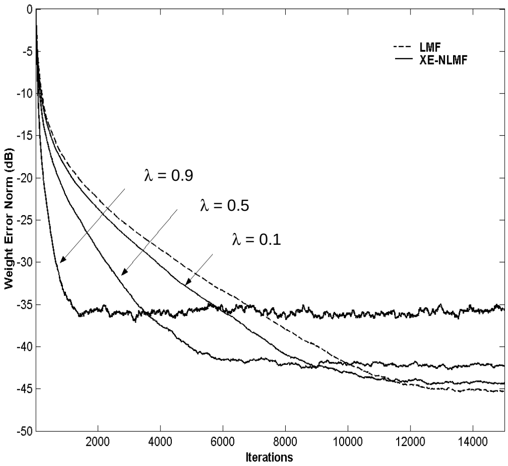


Figure 1. Effect of λ in the XE-NLMF algorithm.

# 4. Simulation Results

In the first experiment, a system identification set up is used to evaluate the algorithm's performance. The unknown system is modelled by a $N$=10 time-invariant FIR filter with the following weights:

$\mathbf{w}^*$ = [0.035 -0.068 0.12 -0.258 0.9 -0.25 0.10 -0.07 0.067 -0.067]$^{\mathbf{T}}$.

The input signal $x(n)$ is obtained by passing a white Gaussian noise $u(n)$ through a channel, $x(n) = x(n-1) + 0.6u(n)$; coloured input signal. The signal to noise ratio is set at 20dB. Two types of additive noises ($\eta(n)$) are to be tested, white Gaussian noise and binary additive noise (sub-Gaussian). The performance considered is the normalised weight error norm, $10\log10(\|\mathbf{w}^* - \mathbf{w}(n)\|^2 / \|\mathbf{w}^*\|^2)$. 300 independent runs will average the results. The step size for the XE-NLMF and the proposed variable normalised XE-NLMF algorithm is set to the same value, $\gamma_{xe} = 0.1$. The NLMS step size is $\mu_{nlms} = 0.2$. Other parameters are $\nu = 0.98$ and $a = 0.9$.

In Figure 1, the convergence and steady state error of different λ values affect the XE-NLMF performance, significantly. Thus, a larger value for λ is desired for fast convergence and smaller value for lower steady state error.

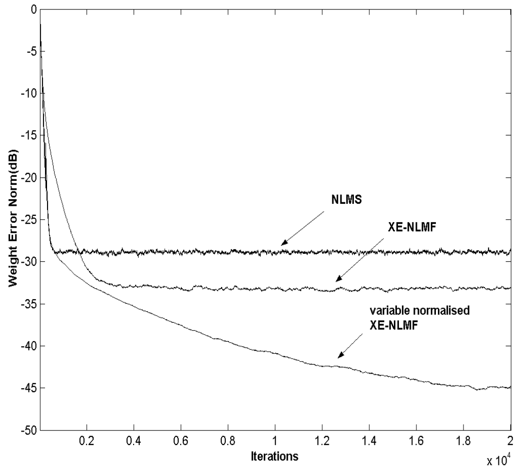


Figure 2. Convergence performance for the proposed algorithm, the XE-NLMF algorithm (λ= 0.9) and the NLMS algorithm in white Gaussian noise.

Figure 2 demonstrates the convergence behaviour of the proposed variable normalised XE-NLMF, the XE-NLMF and the NLMS algorithms under the same convergence rate, in white Gaussian noise. As can be seen in this result that the variable normalised XE-NLMF algorithm adapts faster than the XE-NLMF and NLMS algorithms. At the same time, producing lower steady state weight error norm of more than 15dB. Hence, this has demonstrated the advantages of incorporating a variable mixed-power parameter for further improvement of the XE-NLMF algorithm.

For the Binary additive noise in Figure 3, again it demonstrates the variable normalised XE-NLMF algorithm fast convergence and with lower steady state error. An Improvement of about 25dB in weight error norm over the NLMS algorithm can be seen. This has showed that this LMF-based algorithm is better for sub-Gaussian noise. While, the convergence and steady state error of the NLMS is about the same under Binary noise.

In the second experiment, a channel equaliser is used to determine performance in terms of bit error rate (BER) on the proposed variable normalised XE-NLMF algorithm. The channel equalisation set up will be similar to [6], the channel is $h(z) = 1 + 0.4z^{-1}$ and a co-channel $c(z) = 1 + 0.2z^{-1}$.

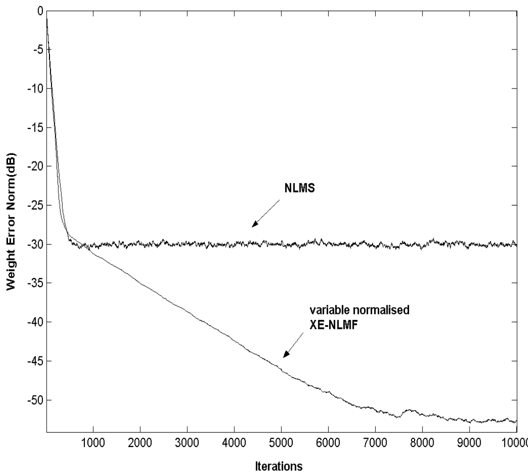


Figure 3. Convergence performance in Binary additive noise.

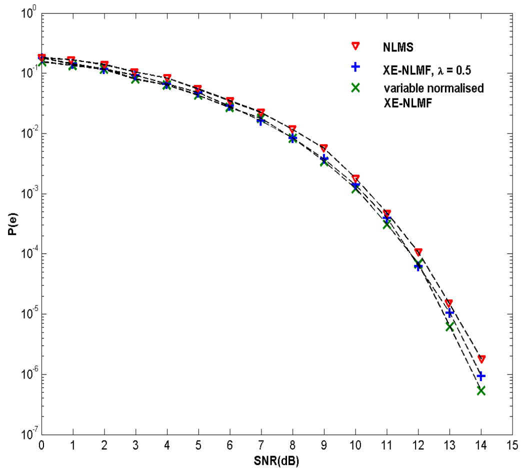


Figure 4. The BER performance in AWGN.

The BER result in AWGN (SNR) is shown in Figure 4 and with co-channel interference (SIR) in Figure 5. This demonstrates that the incorporated variable normalisation does not affect the BER performance. As expected in White Gaussian noise (AWGN) the BER performance is similar for the compared algorithms. In sub-Gaussian noise of co-channel interference, the XE-NLMF and the proposed variable normalised have produce 2dB improvement in BER over the NLMS algorithm.

## 5. Conclusions

This work has proposed a variable normalised XE-NLMF algorithm with a variable mixed-power parameter ($\lambda$). The variable mixed-power parameter follows the scheme of variable step size-LMS and is effective in controlling the mixed-power parameter. This variable normalisation strategy provides an optimised mixed normalisation of signal power and error power for the LMF algorithm. Thus, removing the user selection of the mixed-power parameter and provides extra performance gain from the XE-NLMF algorithm. Under the sub-Gaussian noise, the performance improvement becomes more apparent. Thus, the variable normalised XE-NLMF algorithm is able to produce better adaptive performance in the sub-Gaussian noise and also in Gaussian noise conditions.

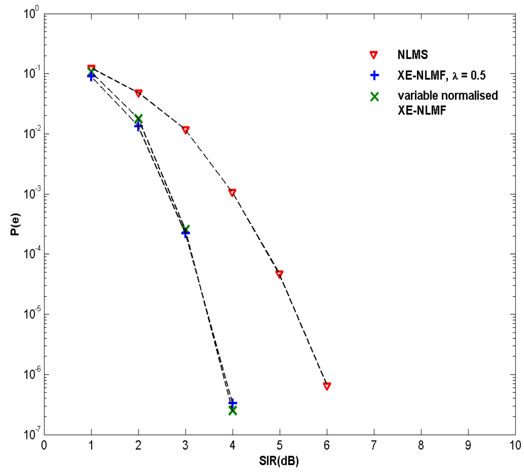


Figure 5. The BER performance in a co-channel interference.

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