A NECESSARY AND SUFFICIENT CONDITION FOR THE BIBO STABILITY OF GENERAL-ORDER BODE-TYPE VARIABLE-AMPLITUDE WAVE-DIGITAL EQUALIZERS

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ABSTRACT

Recently, the authors developed a new synthesis technique for the design of higher-order Bode-type variable-amplitude (*VA*) wave-digital (*WD*) equalizers. The salient feature of the resulting *VA WD* equalizers is that they permit the *continuous* variation of the *WD* equalizer transfer function from a shaping transfer function to its inverse by changing the value of a single variable digital multiplier *only*. The proposed design technique was based on the *WD* realization of the corresponding positive-real analog prototype shaping impedance function, and on the realization of the equalizer transfer function as the reflectance of the shaping impedance function with respect to the constituent variable digital multiplier. This paper is concerned with an investigation of the bounded-input bounded-output (BIBO) stability of *general-order VA WD* equalizers. It is shown that the resulting conditions are both necessary and sufficient for the BIBO stability of the *VA WD* equalizers for the entire range of values for the variable digital multiplier. These conditions can be checked in a straightforward fashion in terms of the characteristics of the shaping transfer function alone. An application example is given to illustrate the main results.

1. INTRODUCTION

In his classical paper more than than six decades ago, Bode [1] introduced the concept of variable-amplitude (*VA*) analog equalizers. These equalizers are capable of varying the gain associated with various frequency bands along the audio signal frequency spectrum, and find practical applications in multimedia, digital audio, digital signal enhancement/correction and hearing aids.

The magnitude-frequency response characteristic of a Bode-type *VA* analog equalizer satisfies a relationship of the form

$$T_v(s) = \frac{r_v + T_s(s)}{1 + r_v T_s(s)}, \tag{1}$$

where $T_v(s)$ is the transfer function of the *VA* equalizer, where $T_s(s)$ is a fixed *shaping* transfer function, and where $s = j\omega$ is the continuous-time (analog) complex frequency-variable. Moreover, $r_v = R_v/R_0$, where R_v is a (positive) variable resistor, and where R_0 is a suitable reference resistance. In this way, the variation of r_v from 0 (via 1) to ∞ results in a *geometrical* variation of $|T_v(s)|$ from $|T_s(j\omega)|$ (via 1) to $|T_s(j\omega)|^{-1}$. Equivalently, the variation of r_v from 0 (via 1) to ∞ results in an *arithmetical* variation of $|T_v(j\omega)|_{dB}$ from $|T_s(j\omega)|_{dB}$ (via $0dB$) to $-|T_s(j\omega)|_{dB}$.

In a previous paper [2], the authors developed a synthesis technique for the design of *higher-order* Bode-type *VA* wave-digital (*WD*) [3] equalizers consisting of one single variable digital multiplier *only*. This design technique was based on the derivation of a corresponding positive-real analog prototype shaping impedance function, on the *WD* realization of the prototype shaping impedance function (employing the bilinear analog-to-digital frequency transformation), and on the realization of the desired equalizer transfer function as the reflectance of the shaping impedance function with respect to the constituent variable digital multiplier.

This paper is concerned with an investigation of the bounded-input bounded-output (BIBO) stability of *general-order VA WD* equalizers. These investigations are carried out in terms of the analog prototype equalizer transfer function by recognizing the fact that the bilinear analog-to-digital frequency transformation preserves the BIBO stability or instability in the corresponding *VA WD* equalizer.

2. THEORETICAL BACKGROUND

Let us consider a continuous-time analog prototype shaping transfer function $T_s(s)$ satisfying the following constrants:

Constraint 1: The analog prototype shaping transfer function $T_s(s)$ is a real rational function of s (in order for $T_s(s)$ to be realizable).

Constraint 2: a) $0 < |T_s(j\omega)| \leq 1$ (used in what follows), or b) $1 \leq |T_s(j\omega)| < \infty$ for all ω (in order for the sign of $|T_v(j\omega)|_{dB}$ remain unchanged as r_v varies in the "half-interval" 0 to 1[1]).

Theorem 1: In order for the analog prototype equalizer transfer function $T_v(s)$ to be a BIBO stable transfer function for all values of the variable resistor r_v, the shaping transfer function $T_s(s)$ must be a minimum-phase function.

Proof: Since $T_v(s) = T_s(s)$ for $r_v = 0$, and since $T_v(s) = 1/T_s(s)$ for $r_v = \infty$ (c.f. Eqn. 1), in order for $T_v(s)$ to be a BIBO stable transfer function, it is necessary that both $T_s(s)$ and $1/T_s(s)$ be BIBO stable transfer functions. This implies that the shping transfer function $T_s(s)$ must be a minimum-phase function.

Theorem 2: $T_s(s)$ must be also a strictly minimum phase transfer function.

Proof: Since $|T_s(j\omega)|$ and $|1/T_s(j\omega)|$ are both bounded for all values of ω (c.f. Constraint 2), $T_s(s)$ must be devoid of both poles and zeros on the imaginary axis of the complex s-plane. Consequently, the minimum-phase transfer function $T_s(s)$ (c.f. Theorem 1) must also be a strictly minimum-phase function.

[1] Due to its *symmetrical* variation, sign of $|T_v(j\omega)|_{dB}$ also remains unchanged when r_v varies in the other "half-interval", i.e. from 1 to ∞.

Let us assume that the shaping transfer function $T_s(s)$ is of general order n, having a magnitude-response characteristic which satisfies the high-level system design specifications

$$q^{-1} \leq |T_s(j\omega)| \quad \leq 1 \quad \text{for} \quad \omega \in \Omega_p, \tag{2}$$
$$h^{-1} \leq |T_s(j\omega)| \leq p^{-1} \quad \text{for} \quad \omega \in \Omega_a, \tag{3}$$

where Ω_p (Ω_a) represents the passband (stopband) frequency region(s) of the transfer function $T_s(s)$, and where $h > p > q > 1$ are equalizer design parameters. Moreover, let us define a pair of strictly BIBO stable real rational continuous-time transfer functions $T_m(s)$ of orders n_m in accordance with

$$T_m(s) = \frac{N_m(s)}{D_m(s)} \quad \text{for} \quad m = 1, 2, \tag{4}$$

where $N_m(s)$ represents the numerator polynomial, and where $D_m(s)$ represents the denominator polynomial of $T_m(s)$. In addition, let the transfer functions $T_m(s)$ possess a pair of magnitude-response characteristics satisfying the system design specifications

$$0 \leq -|T_m(j\omega)|_{dB} \leq A_{pm} \quad \text{for} \quad \omega \in \Omega_{pm}, \tag{5}$$
$$-|T_m(j\omega)|_{dB} \geq A_{am} \quad \text{for} \quad \omega \in \Omega_{am}, \tag{6}$$

in terms of passband ripples (stopband losses) A_{pm} (A_{am}), and in terms of passband (stopband) frequency region(s) Ω_{pm} (Ω_{am}). Finally, let the transfer functions $T_m(s)$ have the same orders $n_1 = n_2 \equiv n$, and have the same passband (stopband) frequency region(s) $\Omega_{p1} = \Omega_{p2} \equiv \Omega_a$ ($\Omega_{a1} = \Omega_{a2} \equiv \Omega_p$).

Theorem 3: The shaping transfer function $T_s(s)$ satisfying the above specifications can be derived as

$$T_s(s) = \frac{1}{h}\frac{T_2(s)}{T_1(s)}, \tag{7}$$

provided that

$$A_{p1} = 10\log\frac{h^2-1}{p^2-1} \quad \text{and} \quad A_{a1} = 10\log\frac{h^2-1}{q^2-1}, \tag{8}$$

and

$$A_{p2} = 10\log\frac{p^2\left(h^2-1\right)}{h^2\left(p^2-1\right)} \quad \text{and} \quad A_{a2} = 10\log\frac{q^2\left(h^2-1\right)}{h^2\left(q^2-1\right)}. \tag{9}$$

Proof: See [2].

The transfer functions $T_m(s)$ will have the same numerator polynomials (to within multiplication by a constant) in accordance with

$$N_2(s) = hN_1(s) \equiv N(s). \tag{10}$$

Therefore, from Eqns. 10, 7 and 4,

$$T_s(s) = D_1(s)/D_2(s), \tag{11}$$

rendering $T_s(s)$ as a strictly minimum-phase transfer function (c.f. Theorem 2).

3. *VA WD* EQUALIZER REALIZATION

Let us define a normalized driving-point impedance function $\hat{Z}(s)$ in accordance with

$$\hat{Z}(s) = \frac{T_s(s)-1}{T_s(s)+1}, \tag{12}$$

Theorem 4: $\hat{Z}(s)$ in Eqn. 12 is a positive-real impedance function.
Proof: According to Talbot [4], $\hat{Z}(s)$ is a positive-real impedance function since $T_s(s)$ has no poles in the right-half of the complex s-plane (c.f. Theorem 2), and since $|T_s(j\omega)|$ is bounded for all ω (c.f. Constraint 2).

In terms of Brune's sections, the driving-point impedance function $\hat{Z}(s)$ can be realized as shown in Fig. 1, where the leftmost parallel tuned section realizes possible zeros at $s = 0$ and $s = \infty$, and where the remaining part realizes a *minimum* impedance function [4]. In this way, by using Eqn. 12, $T_s(s)$ can be expressed in terms of $\hat{Z}(s)$ in accordance with

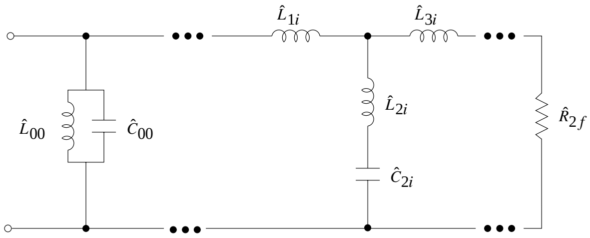

Figure 1: The normalized driving-point impedance function $\hat{Z}(s)$.

$$T_s(s) = \frac{1+\hat{Z}(s)}{1-\hat{Z}(s)}. \tag{13}$$

Then, by invoking Eqn. 13 in Eqn. 1, and by manipulating the resulting equation, one obtains

$$T_v(s) = -\frac{\hat{Z}(s)-\hat{r}_{1v}}{\hat{Z}(s)+\hat{r}_{1v}}, \tag{14}$$

where

$$\hat{r}_{1v} = \frac{1+r_v}{1-r_v} \tag{15}$$

Therefore, the transfer function $T_v(s)$ can be realized as the reflectance of $\hat{Z}(s)$ with respect to $\hat{r}_{1v}$, leading to the desired *VA WD* equalizer as shown Fig. 2. The resulting *WD* equalizer realizes a transfer function

$$T_{v,WD}(z) = B_2(z)/E(z), \tag{16}$$

where $T_{v,WD}(z)$ is related to $T_v(s)$ through the bilinear analog-to-digital frequency transformation $s = 2f_s\frac{z-1}{z+1}$, where z represents the discrete-time (digital) complex-frequency variable, and where f_s represents the sampling frequency.

The *VA WD* realization in Fig. 2 consists (from left to right) of a conventional parallel two-port adaptor [5] and a chain of two-port subnetworks, with each subnetwork being easily identified by using the corresponding realizations in Fig. 3 or their duals in Fig. 4 [6]. In this *VA WD* equalizer realization, the desired variations in $\hat{r}_{1v}$ can be implemented directly through the variation of the value of the single digital multiplier m_v within the leftmost two-port parallel adapter [2].

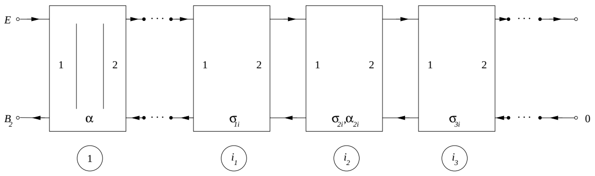

Figure 2: The desired *VA WD* equalizer realization.

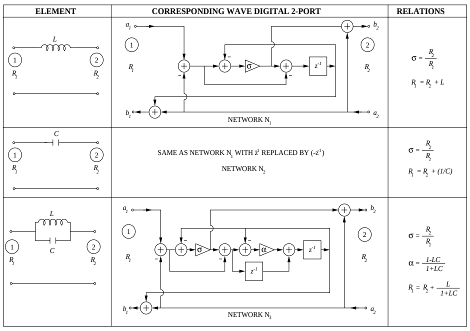

Figure 3: *WD* realizations for various two-port subnetworks.

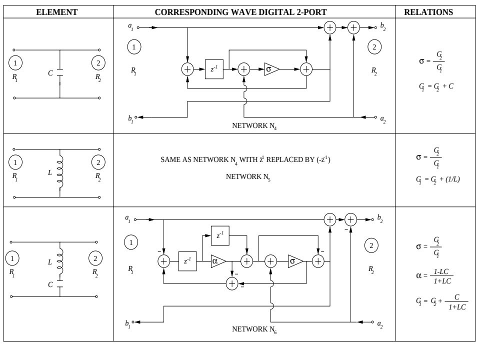

Figure 4: *WD* realizations for various dual two-port subnetworks.

4. BIBO STABILITY INVESTIGATION FOR THE RESULTING *VA WD* EQUALIZER

It is well known that the bilinear frequency transformation preserves the BIBO stability or instability of the analog prototype equalizer transfer function $T_v(s)$ in the resulting *VA WD* equalizer transfer function $T_{v,WD}(z)$. Therefore, without any loss in generality, the investigations of BIBO stability can be carried out in terms of the analog prototype equalizer transfer function $T_v(s)$.

The above BIBO stability was investigated in Theorems 1 and 2 for the extreme values $r_v = 0$ and $r_v = \infty$ of the variable resistor r_v. Therefore, it remains to investigate the BIBO stability for the intermediate values $0 < r_v < \infty$ of r_v. From Eqns. 11 and 1,

$$T_v(s) = \frac{D_1(s) + r_v D_2(s)}{D_2(s) + r_v D_1(s)} \qquad (17)$$

Therefor, the BIBO stability of $T_v(s)$ requires the examination of the roots of the characteristic equation

$$K(s) = D_2(s) + r_v D_1(s) = 0. \qquad (18)$$

Let us express $T_s(s)$ in the general form

$$T_s(s) = \frac{d_{10} + d_{11}s^1 + \ldots + d_{1n}s^n}{d_{20} + d_{21}s^1 + \ldots + d_{2n}s^n}, \qquad (19)$$

where the coefficients d_{mi} (for $i = 1, 2, ..., n$) are (or can be made) non-negative[2].

Theorem 5: The Characteristic equation $K(s)$ in Eqn. 18 is degree invariant for all values of $0 < r_v < \infty$.

Proof: From Eqns. 19 and 18, $K(s)$ will be degree-invariant provided that

$$d_{2n} + r_v d_{1n} \neq 0. \qquad (20)$$

The proof is at once established by recognizing the fact that at least one of the two (non-negative) coefficients d_{2n} or d_{1n} is non-zero, and by recalling that $0 < r_v < \infty$.

Theorem 6: In order for $T_v(s)$ to be BIBO stable for all (intermediate) values $0 < r_v < \infty$ of the variable resistor r_v, it is sufficient for $T s(s)$ to be a strictly minimum-phase transfer function.

Proof: If $K(s)$ is degree-invariant, then the roots of $K(s)$ will vary continuously along a constant number of root-locus branches [7]. The transfer function $T_v(s)$ will become BIBO unstable if and only if one or more of the root-locus branches associated with $K(s)$ intersects the imaginary axis of the complex s-plane, i.e. if

$$1 + r_v T_s(j\omega_0) = 0 \qquad (21)$$

for one or more frequencies ω_0. Since r_v is a real-valued resistance, Eqn. 21 is equivalent to

$$\Im\{T_s(j\omega_0)\} = 0 \text{ and } 1 + r_v \Re\{T_s(j\omega_0)\} = 0, \qquad (22)$$

where $\Re\{.\}$ and $\Im\{.\}$ represent the real and imaginary parts of $\{.\}$, respectively.

In this way, the transfer function $T_v(s)$ in Eqn. 17 will be BIBO stable for all values of $0 < r_v < \infty$ provided that,

$$\text{when } \Im\{T_s(j\omega_0)\} = 0, \text{ then } 1 + r_v \Re\{T_s(j\omega_0)\} > 0. \qquad (23)$$

The real and imaginary parts of $T_s(j\omega)$ can be conveniently be obtained as

$$\Re\{T_s(j\omega)\} = \frac{\Re\{D_1(j\omega)\}\Re\{D_2(j\omega)\} + \Im\{D_1(j\omega)\}\Im\{D_2(j\omega)\}}{\Re\{D_2(j\omega)\}^2 + \Im\{D_2(j\omega)\}^2}, \qquad (24)$$

and

$$\Im\{T_s(j\omega)\} = \frac{-\Re\{D_1(j\omega)\}\Im\{D_2(j\omega)\} + \Im\{D_1(j\omega)\}\Re\{D_2(j\omega)\}}{\Re\{D_2(j\omega)\}^2 + \Im\{D_2(j\omega)\}^2}. \qquad (25)$$

From Eqn. 25, when $\Im\{T_s(j\omega_0)\} = 0$,

$$\Re\{D_1(j\omega_0)\}\Im\{D_2(j\omega_0)\} + \Im\{D_1(j\omega_0)\}\Re\{D_2(j\omega_0)\} = 0. \qquad (26)$$

Subsequently, from Eqns. 26 and 24, when $\Im\{T_s(j\omega_0)\} = 0$,

$$\Re\{T_s(j\omega_0)\} = \Re\{D_1(j\omega_0)\}\Re\{D_2(j\omega_0)\} + \frac{\Re\{D_1(j\omega_0)\}}{\Re\{D_2(j\omega_0)\}}\Im\{D_2(j\omega_0)\}^2. \qquad (27)$$

[2] This is due to the fact that $D_1(s)$ and $D_2(s)$ are strictly Hurwits polynomials.

But, $\Re\{D_1(j\omega)\} > 0$ and $\Re\{D_2(j\omega)\} > 0$ for all values of ω (since $D_1(s)$ and $D_2(s)$ are strictly Hurwitz polynomials). Consequently, when $\Im\{T_s(j\omega_0)\} = 0$, $\Re\{T_s(j\omega_0)\} > 0$, giving rise to satisfaction of Eqn. 23

Theorem 7: Subject to the satisfaction of Constraints 1 and 2, it is both necessary and sufficient for the analog prototype shaping transfer function $T_s(s)$ to be a strictly minimum-phase transfer function if it is required that the corresponding $VA\ WD$ equalizer transfer function $T_{vWD}(z)$ be BIBO stable for all values of the constituent variable digital multiplier m_v.

Proof: The proof is at once established by taking into account Theorems 3 and 6 together with the properties of the bilinear frequency transformation (as pointed out before).

5. APPLICATION EXAMPLE

Let us consider the design of a fourth-order bandpass elliptic *VA WD* equalizer satisfying the following specifications.

$$h = 4.00 \qquad p = 3.5 \qquad q = 1.5$$
$$f_{p1} = 8.00\ Hz \quad f_{p2} = 10.00\ Hz$$
$$f_{a1} = 6.00\ Hz \quad f_{a2} = 12.00\ Hz \quad 1/T = 32.00\ Hz$$

Then, $A_{p1} = 1.249387366\ dB$, $A_{a1} = 28.753975887\ dB$, and $A_{p2} = 0.089548426\ dB$, $A_{a2} = 16.798711857\ dB$. Consequently,

$$d_{10} = 0.99016629597150 \qquad d_{20} = 1.0$$
$$d_{11} = 30.78043908024169 \qquad d_{21} = 71.50269599061528$$
$$d_{12} = 13160.26562996701 \qquad d_{22} = 15832.27752922151$$
$$d_{13} = 188687.0835399217 \qquad d_{23} = 438318.4767617985$$
$$d_{14} = 37208559.89697220 \qquad d_{24} = 3757809.172899097$$

Then, the variation of the magnitude-frequency response of the corresponding *VA WD* equalizer as a function of the variable resistor r_v can be obtained as shown in Fig. 5. From Fig. 5, the $VA\ WD$ equalizer magnitude-frequency response exhibits the desirable arithmetically symmetric variations (around 0 dB) for geometrically symmetric variations in r_v values.

Finally, the root locus branches for the poles p_i (for $i = 1, 2, 3, 4$) of the analog prototype equalizer transfer function $T_v(s)$ as a function of the variable resistor r_v are as shown in Fig. 6. From Fig. 6, the poles of $T_v(s)$ remain in the left half of the complex s-plane, rendering $T_v(s)$ as BIBO stable for the entire range of values $0 \leq r_v \leq$ of the variable resistor r_v.

6. ACKNOWLEDGEMENTS

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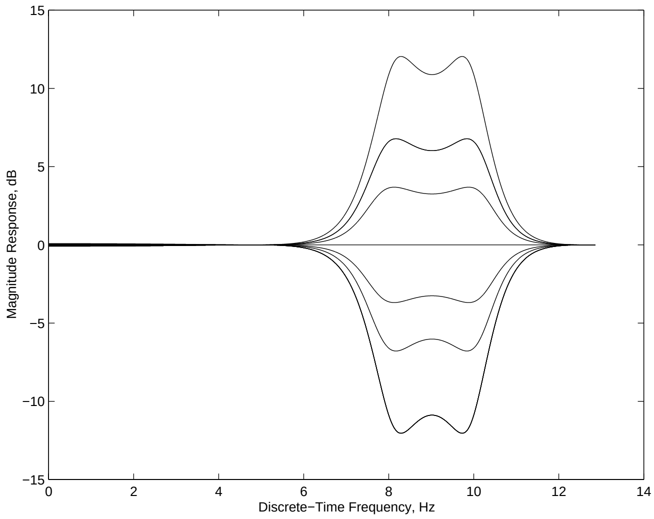

Figure 5: *VA WD* equalizer magnitude-frequency response for $r_v = 0, 0.25, 0.5, 1.0, 2.0, 5.0, \infty$from bottom to top.

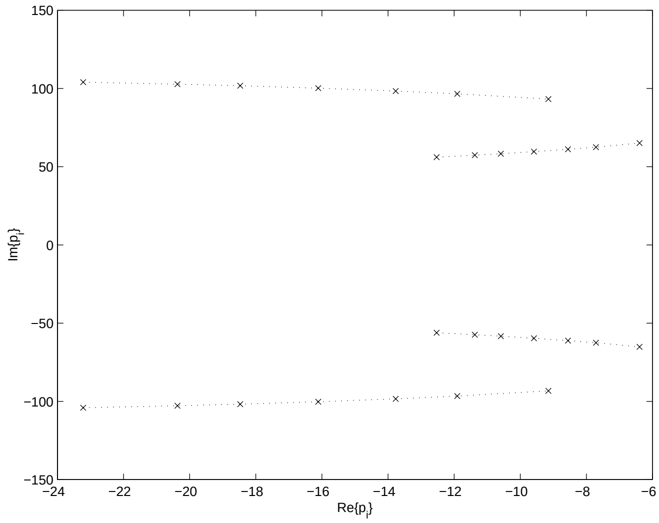

Figure 6: *VA WD* Root locus branches for $r_v = 0, 0.25, 0.5, 1.0, 2.0, 5.0, \infty$.

7. REFERENCES

[1] H.W. Bode, "Variable Equalizers", *Bell Syst. Tech. J.*, vol. 17, pp. 229-244, April 1938.

[2] B. Nowrouzian, A.T.G. Fuller, and M.N.S. Swamy, "An Alternative Approach to the Design and Synthesis of Higher-Order Bode-Type Variable-Amplitude Wave-Digital Equalizers Equalizers" *Proceedings of the 2002 IEEE International Symposium on Circuits and Systems*, Scottsdale, Arisona, U.S.A., May 26-29, 2002.

[3] A. Fettweis, "Wave Digital Filters:Theory and Practise", *Proc. IEEE*, vol. 73, no. 2, p. 270, February 1988.

[4] M. E. Van Valkenburg, *Introduction to Modern Network Synthesis*, John Wiley & Sons, Inc., New York, 1960.

[5] A. Antoniou, "DIGITAL FILTERS Analysis, Design, and Applications", *McGraw-Hill Inc.*, New York, 1993.

[6] M.N.S. Swamy and K.S. Thyagarajan, "A New Type of Wave Digital Filter", *Journal of Franklin Institute*, Vol. 300, pp. 41-58, July 1975

[7] B.R. Barmish, "New Tools for Robustness of Linear Systems", *Macmillian Publishing Company*, New York, 1994.