ANALYSIS OF ADAPTIVE FILTER ALGORITHMS IN FEEDBACK SYSTEMS

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ABSTRACT

This paper presents some convergence results of an adaptive filter algorithm for the feedback-type active noise control (ANC) system with feedback path modeling using a dither signal and that of hearing aids. In these systems there is an adaptive filter in the feedback path so that the conventional method for analysis does not work. A frequency domain method is used for the analysis. It is based on the averaging method and the frequency domain expression of the adaptive algorithm with the causality constraint. The convergence conditions are derived for both problems and a bias expression of the weights for the latter problem is presented. The positive realness of the transfer function of the optimal predictor of the incoming signal is required for stability. Finally, the convergence condition and the bias are explicitly given for simple examples and their validities are shown by some simulations.

1. INTRODUCTION

The adaptive filter algorithm in the feedback-type active noise control (ANC) system has been first analyzed in [1]. In this system an adaptive filter is operating in the feedback path so that the conventional method for analysis does not work. In [1] a method has been devised which is a combination of the averaging method and a frequency domain expression of the algorithm to obtain a local stability (convergence) condition. As is well known in the feedforward-type ANC system, the convergence is guaranteed by the so-called 90° condition [2]. The corresponding condition has been derived in [1]. It depends on the optimal prediction filter of the incoming noise to be reduced as well as the transfer functions of the feedback path and its estimate.

On the other hand, several schemes have been proposed in the feedforward ANC system to estimate the secondary path on-line by injecting a dither signal [2]. In this paper, a similar scheme is considered but in this system two adaptive filters are operating in the feedback paths so that the stability of the total system is not obvious. By applying the above frequency domain technique the local stability condition is derived in terms of the prediction error filter of the noise.

As for hearing aids, there is an acoustic feedback path from the receiver to the microphone and this causes annoying effects such as whistling and howling. An adaptive filter is used to model the acoustic feedback path. But, again it is in the feedback path so that the stationary point of the adaptive filter is biased from the true feedback path transfer function. In [3] based on the time domain approach, an approximate expression of this bias in the weight vector of the adaptive filter has been derived by assuming that it is very small. Here, again via the frequency domain approach we derive a corresponding expression of the bias in terms of the prediction error filter of the incoming signal to the hearing-aid. The stability condition is also derived.

2. ANALYSIS OF THE FEEDBACK-TYPE ANC WITH ON-LINE MODELING OF THE FEEDBACK PATH

In the feedback-type ANC system the incoming noise is picked at the microphone and from the loud speaker a control signal is added to reduce the noise level. Also, for on-line modeling of the feedback path transfer function, a dither signal is injected. A block diagram of the feedback-type ANC system in [2] which was analyzed in [1] is shown in Fig.1 where $d(n)$ is a stationary noise with zero mean and the spectral density $P(z)$ and $S(z)$ is the transfer function of the feedback path. The adaptive filter denoted by $W(z)$

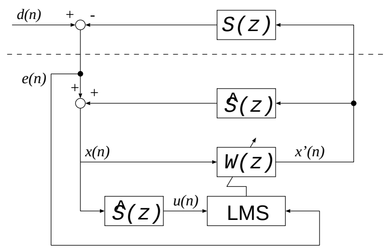

Fig. 1. Block diagram of the feedback-type ANC system.

is used for prediction of $d(n)$ through $S(z)$ but the fixed estimate $\hat{S}(z)$ of $S(z)$ is used for updating the weights of the adaptive filter via the Filtered-X LMS algorithm. Fig.2 shows the block diagram of the ANC system treated in this paper where an additional adaptive filter denoted by $\hat{S}(z)$ is introduced for on-line modeling of $S(z)$ with a dither signal $v(n)$ which is an artificially generated white noise with zero mean and the variance σ_v^2. The weights of the adaptive filters $W(z)$ and $\hat{S}(z)$ are updated by the Filtered-X LMS algorithm and the LMS algorithm, respectively.

The signals $d(n), e(n), e_s(n), e_w(n), x(n), x'(n), u(n), v(n)$ are defined in Fig.2 and the weights of the adaptive filters $W(z), \hat{S}(z)$ are $\{w_i(n)\}(i = 0, \cdots, N_w - 1), \{\hat{s}_i(n)\}(i =$

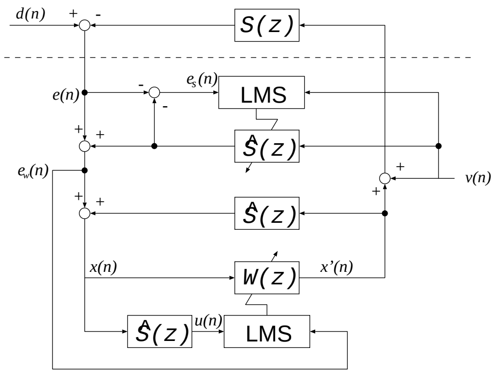

Fig. 2. Block diagram of the feedback-type ANC system with on-line modeling of the feedback path.

$0, \cdots, N_s-1$), respectively. The tap weight vectors are updated by the LMS-type algorithm as

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu_w \boldsymbol{u}(n) e_w(n) \quad (1)$$
$$\hat{\boldsymbol{s}}(n+1) = \hat{\boldsymbol{s}}(n) + \mu_s \boldsymbol{v}(n) e_s(n) \quad (2)$$

where N-dimensional vectors $\boldsymbol{w}(n), \hat{\boldsymbol{s}}(n), \boldsymbol{u}(n)$ are defined by

$$\begin{aligned} \boldsymbol{w}(n) &= [w_0(n), \cdots, w_{N_w-1}(n), 0, \cdots, 0]^T \\ \hat{\boldsymbol{s}}(n) &= [\hat{s}_0(n), \cdots, \hat{s}_{N_s-1}(n), 0, \cdots, 0]^T \\ \boldsymbol{u}(n) &= [u(n), \cdots, u(n-N_w-1), 0, \cdots, 0]^T \end{aligned} \quad (3)$$

and $\boldsymbol{x}(n), \boldsymbol{x'}(n), \boldsymbol{e}(n), \boldsymbol{d}(n), \boldsymbol{v}(n)$ are defined similarly. Also, $\boldsymbol{s}$ is a tap weight vector corresponding to $S(z)$. Padding some zeros in (3) is required to keep the causality of the adaptive filters $W(z), \hat{S}(z)$ when the analysis is done in the frequency domain and N is taken as $N \geq 2\max(N_w, N_s)$[1]. From Fig.2 we have

$$e_w(n) = -e_s(n) = d(n) - \boldsymbol{s} \otimes \boldsymbol{x'}(n) + (\hat{\boldsymbol{s}}(n) - \boldsymbol{s}) \otimes \boldsymbol{v}(n) \quad (4)$$

where "$\otimes$" denotes the convolution. Since $\boldsymbol{x'}(n) = \boldsymbol{w}(n) \otimes \boldsymbol{x}(n)$,

$$e_w(n) \simeq d(n) - \boldsymbol{x}^\dagger(n)(\boldsymbol{s} \otimes \boldsymbol{w}(n)) + \boldsymbol{v}^\dagger(n) \Delta \boldsymbol{s}(n) \quad (5)$$

where "$\dagger$" denotes the complex conjugate transpose and $\Delta \boldsymbol{s}(n) = \hat{\boldsymbol{s}}(n) - \boldsymbol{s}$. Let the $N \times N$ DFT (discrete Fourier transform) matrix be

$$\boldsymbol{F} = \left[\exp\left(-j \frac{2\pi k l}{N} \right) \right] \qquad k, l = 0, 1, \cdots, N-1$$

and the N-point DFTs of $\boldsymbol{w}, \boldsymbol{s}, \hat{\boldsymbol{s}}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{x}, \boldsymbol{x'}, \boldsymbol{u}, \boldsymbol{v}$ are denoted by the corresponding capital letters as $\boldsymbol{W}, \boldsymbol{S}, \hat{\boldsymbol{S}}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{X}, \boldsymbol{X'}, \boldsymbol{U}, \boldsymbol{V}$, that is, $\boldsymbol{W} = \boldsymbol{F}\boldsymbol{w}$ etc. Noting the identity $\boldsymbol{F}^\dagger \boldsymbol{F} = N\boldsymbol{I}$, from (5) we have

$$e_w(n) \simeq d(n) - \frac{1}{N} \boldsymbol{X}^\dagger(n) \boldsymbol{\Lambda}_S \boldsymbol{W}(n) + \frac{1}{N} \boldsymbol{V}^\dagger(n) \Delta \boldsymbol{S}(n) \quad (6)$$

where $\boldsymbol{\Lambda}_S$ is a diagonal matrix given by

$$\boldsymbol{\Lambda}_S = \mathrm{diag}[S_0, S_1, \cdots, S_{N-1}] \quad (7)$$

where S_i is the i-th element of $\boldsymbol{S}$. Similarly $\boldsymbol{\Lambda}_{W(n)}, \boldsymbol{\Lambda}_{\hat{S}}$ are defined. From Fig.2 other signals are expressed as

$$\begin{aligned} \boldsymbol{E}(n) &= \boldsymbol{D}(n) - \boldsymbol{\Lambda}_S^* \left(\boldsymbol{X'}(n) + \boldsymbol{V}(n) \right) \\ \boldsymbol{X'}(n) &= \boldsymbol{\Lambda}_{W(n)}^* \boldsymbol{X}(n), \ \boldsymbol{U}(n) = \boldsymbol{\Lambda}_{\hat{S}(n)}^* \boldsymbol{X}(n) \end{aligned} \quad (8)$$

where "$*$" denotes the complex conjugate. Also, from Fig.2

$$\begin{aligned} \boldsymbol{X}(n) &= \boldsymbol{E}(n) + \boldsymbol{\Lambda}_{\hat{S}(n)}^* \boldsymbol{X'}(n) + \boldsymbol{\Lambda}_{\hat{S}(n)}^* \boldsymbol{V}(n) \\ &= \boldsymbol{D}(n) + \boldsymbol{\Lambda}_{\Delta S(n)}^* \boldsymbol{\Lambda}_{W(n)}^* \boldsymbol{X}(n) + \boldsymbol{\Lambda}_{\Delta S}^* \boldsymbol{V}(n). \end{aligned} \quad (9)$$

Then, $\boldsymbol{X}(n)$ is given by

$$\boldsymbol{X}(n) = \boldsymbol{Q}(n)\boldsymbol{D}(n) + \boldsymbol{Q}(n) \boldsymbol{\Lambda}_{\Delta S}^* \boldsymbol{V}(n) \quad (10)$$

where $\boldsymbol{Q}(n)$ is defined by $\boldsymbol{Q}(n) = \left[\boldsymbol{I} - \boldsymbol{\Lambda}_{W(n)}^* \boldsymbol{\Lambda}_{\Delta S}^* \right]^{-1}$. Applying $\boldsymbol{F}$ to (1) and (2), we have

$$\boldsymbol{W}(n+1) = \boldsymbol{W}(n) + \mu_w \boldsymbol{U}(n) e_w(n) \quad (11)$$
$$\hat{\boldsymbol{S}}(n+1) = \hat{\boldsymbol{S}}(n) + \mu_s \boldsymbol{V}(n) e_s(n). \quad (12)$$

Substituting (6),(10) into (11) and (12),we have

$$\begin{aligned} \boldsymbol{W}(n+1) &= \boldsymbol{W}(n) \\ &+ \mu_w \boldsymbol{\Lambda}_{\hat{S}(n)}^* \left[\boldsymbol{Q}(n)\boldsymbol{D}(n) + \boldsymbol{Q}(n) \boldsymbol{\Lambda}_{\Delta S(n)}^* \boldsymbol{V}(n) \right] \\ &\left(d(n) - \frac{1}{N} \left[\boldsymbol{D}^\dagger(n) \boldsymbol{Q}^\dagger(n) + \boldsymbol{V}^\dagger(n) \boldsymbol{\Lambda}_{\Delta S(n)} \boldsymbol{Q}^\dagger(n) \right] \right. \\ &\left. \boldsymbol{\Lambda}_S \boldsymbol{W}(n) + \frac{1}{N} \boldsymbol{V}^\dagger(n) \Delta \boldsymbol{S}(n) \right) \\ \Delta \boldsymbol{S}(n+1) &= \Delta \boldsymbol{S}(n) + \mu_s \boldsymbol{V}(n) \\ &\times \left(-d(n) + \frac{1}{N} \left[\boldsymbol{D}^\dagger(n) \boldsymbol{Q}^\dagger(n) + \boldsymbol{V}^\dagger(n) \boldsymbol{\Lambda}_{\Delta S(n)} \boldsymbol{Q}^\dagger(n) \right] \right. \\ &\left. \boldsymbol{\Lambda}_S \boldsymbol{W}(n) - \frac{1}{N} \boldsymbol{V}^\dagger(n) \Delta \boldsymbol{S}(n) \right). \end{aligned} \quad (13)$$

Next we take the averages with respect to the stochastic signals $\boldsymbol{D}(n), \boldsymbol{V}(n)$ in (13) by fixing $\boldsymbol{W}(n) = \bar{\boldsymbol{W}}(n), \Delta \boldsymbol{S}(n) = \Delta \bar{\boldsymbol{S}}(n)$. Since $d(n)$ and $v(n)$ are uncorrelated and for large N, the element of $\boldsymbol{D}(n)$ is uncorrelated with each other. From [1] we have

$$\mathrm{E}\left[\boldsymbol{D}(n) \boldsymbol{D}^\dagger(n) \right] \simeq N \mathrm{diag}[P_0, P_1, \cdots, P_{N-1}] = \boldsymbol{\Lambda}_P \quad (14)$$

where $P_i = P(\mathrm{e}^{j\frac{2\pi i}{N}})$ and

$$\mathrm{E}[\boldsymbol{D}(n) d(n)] \simeq [P_0, P_1, \cdots, P_{N-1}]^T = \boldsymbol{P} \quad (15)$$
$$\mathrm{E}\left[\boldsymbol{V}(n) \boldsymbol{V}^\dagger(n) \right] \simeq N \sigma_v^2 \boldsymbol{I}. \quad (16)$$

Hence, the averaged system of (13) is described as

$$\begin{aligned} \bar{\boldsymbol{W}}(n+1) &= \bar{\boldsymbol{W}}(n) + \mu_w \left[\boldsymbol{\Lambda}_{\hat{\bar{S}}(n)}^* \bar{\boldsymbol{Q}}(n) \boldsymbol{P} \right. \\ &\quad - \boldsymbol{\Lambda}_{\hat{\bar{S}}(n)}^* \bar{\boldsymbol{Q}}(n) \boldsymbol{\Lambda}_P \bar{\boldsymbol{Q}}^\dagger(n) \boldsymbol{\Lambda}_S \bar{\boldsymbol{W}}(n) \\ &\quad + \boldsymbol{\Lambda}_{\hat{\bar{S}}(n)}^* \bar{\boldsymbol{Q}}(n) \boldsymbol{\Lambda}_{\Delta \bar{S}(n)}^* \sigma_v^2 \\ &\quad \left. \times \left[-\boldsymbol{\Lambda}_{\Delta \bar{S}(n)} \bar{\boldsymbol{Q}}^\dagger(n) \boldsymbol{\Lambda}_S \bar{\boldsymbol{W}}(n) + \Delta \bar{\boldsymbol{S}}(n) \right] \right]_+ \\ \Delta \bar{\boldsymbol{S}}(n+1) &= \Delta \bar{\boldsymbol{S}}(n) \\ &\quad + \mu_s \sigma_v^2 \left[\boldsymbol{\Lambda}_{\Delta \bar{S}(n)} \bar{\boldsymbol{Q}}^\dagger(n) \boldsymbol{\Lambda}_S \bar{\boldsymbol{W}}(n) - \Delta \bar{\boldsymbol{S}}(n) \right]_+ \end{aligned} \quad (17)$$

where $\bar{\boldsymbol{Q}}$ is defined by replacing $\boldsymbol{W}(n), \Delta\boldsymbol{S}(n)$ with $\bar{\boldsymbol{W}}(n)$, $\Delta\bar{\boldsymbol{S}}(n)$ in $\boldsymbol{Q}$.The operation $[\,]_+$ means that the causal part is taken from the inverse transform of the quantity in square brackets. This operation is necessary to keep $\bar{\boldsymbol{W}}(n)$ and $\Delta\bar{\boldsymbol{S}}(n)$ causal. Since all the matrices in (17) are diagonal, we can focus on the l-th element as

$$\begin{aligned}
\bar{W}_l(n+1) &= \bar{W}_l(n) + \mu_w \Bigg[\frac{\bar{\hat{S}}_l^*(n) P_l}{1 - \bar{W}_l^*(n)\Delta\bar{S}_l^*(n)} \\
&\quad - \frac{\bar{\hat{S}}_l^*(n) P_l S_l \bar{W}_l(n)}{\left(1 - \bar{W}_l^*(n)\Delta\bar{S}_l^*(n)\right)\left(1 - \bar{W}_l(n)\Delta\bar{S}_l(n)\right)} \\
&\quad + \frac{\bar{\hat{S}}_l^*(n)\Delta\bar{S}_l^*(n)}{1 - \bar{W}_l^*(n)\Delta\bar{S}_l^*(n)}\sigma_v^2 \\
&\quad \times \left(-\frac{\Delta\bar{S}_l(n) S_l \bar{W}_l(n)}{1 - \bar{W}_l(n)\Delta\bar{S}_l(n)} + \Delta\bar{S}_l(n) \right) \Bigg]_+ \\
\Delta\bar{S}_l(n+1) &= \Delta\bar{S}_l(n) \\
&\quad + \mu_s\sigma_v^2 \left[\frac{\Delta\bar{S}_l(n) S_l \bar{W}_l(n)}{1 - \bar{W}_l(n)\Delta\bar{S}_l(n)} - \Delta\bar{S}_l(n) \right]. \quad (18)
\end{aligned}$$

Obviously $\Delta\bar{S}_l = 0$ is a stationary point of (18). By substituting this into (18) the stationary point $\bar{W}_l(n+1) = \bar{W}_l(n) = W_l$ is obtained. As $N \to \infty$, we can replace the discrete frequencies with the continuous ones so that instead of W_l, for example, we use $W(z)$ where $z = e^{j\omega}$. Hence, the stationary point $W(z) = W_{\text{opt}}(z)$ is given by

$$\left[S(z^{-1})P(z) - S(z^{-1})P(z)S(z)W(z)\right]_+ = 0. \quad (19)$$

Let the spectral factorization of $P(z)$ be $P(z) = R(z)R(z^{-1})$ where $R(z)$ is of minimum phase. Also, we assume that $S(z) = z^{-q}C(z)$ where q is the delay and $C(z)$ is a stable polynomial. Then, the solution of (19) is

$$W_{\text{opt}}(z) = A(z)/C(z) \quad (20)$$

where $A(z) = [z^q R(z)]_+ / R(z)$. We note that $A(z)$ is the transfer function of the optimal q-step ahead linear predictor of $d(n)$.

To examine the stability around this stationary point, we calculate the following derivative matrix

$$\begin{pmatrix} \dfrac{\partial \bar{W}_l(n+1)}{\partial \bar{W}_l(n)} & \dfrac{\partial \bar{W}_l(n+1)}{\partial \Delta\bar{S}_l(n)} \\ \dfrac{\partial \Delta\bar{S}_l(n+1)}{\partial \bar{W}_l(n)} & \dfrac{\partial \Delta\bar{S}_l(n+1)}{\partial \Delta\bar{S}_l(n)} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

at $\Delta\bar{S}_l(n) = 0$ and $\bar{W}_l(n) = W_{\text{opt},l}$. We use the same rule of the differentiation in [1] where we can discard the operation $[\,]_+$. Then,

$$\begin{aligned}
J_{11} &= 1 + \mu_w \frac{1}{\left|1 - \bar{W}_l(n)\Delta\bar{S}_l(n)\right|^2} \\
&\quad \times \left[-\frac{\bar{\hat{S}}_l^*(n) P_l S_l + \bar{\hat{S}}_l^*(n)\Delta\bar{S}_l^*(n)\Delta\bar{S}_l(n) S_l \sigma_v^2}{1 - \bar{W}_l^*(n)\Delta\bar{S}_l^*(n)} \right] \\
J_{22} &= 1 + \mu_s\sigma_v^2 \\
&\quad \times \left[\frac{S_l \bar{W}_l^2(n)\Delta\bar{S}_l(n)}{\left|1 - \bar{W}_l(n)\Delta\bar{S}_l(n)\right|^2} + \frac{S_l \bar{W}_l(n)}{\left(1 - \bar{W}_l(n)\Delta\bar{S}_l(n)\right)} - 1 \right].
\end{aligned}$$

At the stationary point we have

$$J_{11,\text{opt}} = 1 - \mu_w S_l^* P_l S_l \quad (21)$$

$$J_{22,\text{opt}} = 1 - \mu_s\sigma_v^2 \left[1 - S_l W_{\text{opt},l}\right] \quad (22)$$

and $J_{21} = 0$. Hence for stability, since μ_w and μ_s are small positive constants and $|S_l|^2 P_l > 0$, $\text{Re}\left[1 - S(z)W_{\text{opt}}(z)\right] > 0$ is required. From (20) this condition is

$$\text{Re}\left[1 - z^{-q}A(z)\right] > 0 \quad (23)$$

with $z = e^{j\omega}$. We note that $1 - z^{-q}A(z)$ is just the transfer function of the optimal q-step ahead linear prediction error filter of $d(n)$.

3. ANALYSIS OF THE ADAPTIVE ALGORITHM IN HEARING AIDS

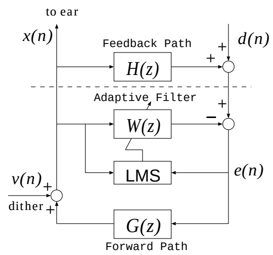

Fig. 3. Block diagram of the hearing aid plant.

Fig.3 shows the block diagram of a hearing aid plant where $d(n)$ is a stationary incoming signal with zero mean and the spectral density $P(z)$ and the forward path and the feedback path transfer functions are $G(z)$, $H(z)$, respectively. A dither signal $v(n)$ is also added. The weights of the adaptive filter denoted by $W(z)$ is updated by the standard LMS algorithm to cancel the effect of $H(z)$. The analysis of the system in Fig.3 can be done in the same way as in Section 2. The weight vector $\boldsymbol{w}(n)$ of the adaptive filter is updated by

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \mu\boldsymbol{x}(n)e(n) \quad (24)$$

where $\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{e}$ are defined like (3). The error signal $e(n)$ is given by

$$e(n) = d(n) + \boldsymbol{h}^\dagger\boldsymbol{x}(n) - \boldsymbol{w}^\dagger\boldsymbol{x}(n) \quad (25)$$

where $\boldsymbol{h}$ is the weight vector corresponding to $H(z)$. The frequency domain expression of $\boldsymbol{e}(n)$ using the DFT matrix is given by

$$\boldsymbol{E}(n) = \boldsymbol{D}(n) + \boldsymbol{\Lambda}_H^*\boldsymbol{X}(n) - \boldsymbol{\Lambda}_{W(n)}^*\boldsymbol{X}(n) \quad (26)$$

where $\boldsymbol{D}(n), \boldsymbol{E}(n), \boldsymbol{X}(n), \boldsymbol{\Lambda}_H, \boldsymbol{\Lambda}_{W(n)}$ are defined as in Section 2. Also, $\boldsymbol{X}(n) \simeq \boldsymbol{\Lambda}_G^*\boldsymbol{E}(n) + \boldsymbol{V}(n)$. Hence we have

$$\boldsymbol{X}(n) \simeq \boldsymbol{Q}(n)\left(\boldsymbol{\Lambda}_G^*\boldsymbol{D}(n) + \boldsymbol{V}(n)\right) \quad (27)$$

with $\boldsymbol{Q}(n) = \left[\boldsymbol{I} + \boldsymbol{\Lambda}_G^*\left(\boldsymbol{\Lambda}_{W(n)}^* - \boldsymbol{\Lambda}_H^*\right)\right]^{-1}$. Applying F to (24) and using (27) we have

$$\begin{aligned}
&\boldsymbol{W}(n+1) = \boldsymbol{W}(n) + \mu\boldsymbol{Q}(n)\left(\boldsymbol{\Lambda}_G^*\boldsymbol{D}(n) + \boldsymbol{V}(n)\right) \cdot \\
&\left[d(n) - \frac{1}{N}\left(\boldsymbol{D}^\dagger(n)\boldsymbol{\Lambda}_G + \boldsymbol{V}^\dagger(n)\right)\boldsymbol{Q}^\dagger(n)\left(\boldsymbol{W}(n) - \boldsymbol{H}\right) \right]
\end{aligned}$$

Using (14)-(16) the corresponding averaged system is given by

$$\bar{\boldsymbol{W}}(n+1) = \bar{\boldsymbol{W}}(n) + \mu \left[\bar{\boldsymbol{Q}}(n)\boldsymbol{\Lambda}_G^*\boldsymbol{P} \right.$$
$$\left. -\bar{\boldsymbol{Q}}(n)\left(\boldsymbol{\Lambda}_G^*\boldsymbol{\Lambda}_P\boldsymbol{\Lambda}_G + \sigma_v^2\boldsymbol{I}\right)\bar{\boldsymbol{Q}}^\dagger(n)\left(\bar{\boldsymbol{W}}(n) - \boldsymbol{H}\right)\right]_+ . \quad (28)$$

The l-th component is written as

$$\bar{W}_l(n+1) = \bar{W}_l(n)$$
$$+\mu \left[\frac{G_l^* P_l - \sigma_v^2\left(\bar{W}_l(n) - H_l\right)}{\left(1 + G_l^*(\bar{W}_l^*(n) - H_l^*)\right)\left(1 + G_l\left(\bar{W}_l(n) - H_l\right)\right)}\right]_+ . \quad (29)$$

At the stationary point $W(z) = W_{\text{opt}}(z)$ we assume that $1 + G(z)(W(z) - H(z))$ is stable. Then from (29) we have

$$\left[\frac{G(z^{-1})P(z) - \sigma_v^2(W(z) - H(z))}{1 + G(z)(W(z) - H(z))}\right]_+ = 0. \quad (30)$$

In general it is difficult to solve this "generalized" Wiener-Hopf equation. We consider the special case of no dither signal, that is, $\sigma_v^2 = 0$, and we assume that $G(z) = z^{-q}G_c(z)$ where $G_c(z)$ is a stable polynomial. Then, (30) becomes

$$[z^q R(z)]_+ = \frac{G_c(z)R(z)B(z)}{1 + G_c(z)z^{-q}B(z)} \quad (31)$$

where we set the bias $B(z)$ as $B(z) = W_{\text{opt}}(z) - H(z)$ and assume that $1 + z^{-q}G_c(z)B(z)$ is stable. Hence,we have

$$B(z) = \frac{A(z)}{G_c(z)(1 - z^{-q}A(z))} \quad (32)$$

where we assume that the prediction error filter $1 - z^{-q}A(z)$ is stable. Taking the derivative of (29) and substituting this stationary point $W_{\text{opt},l} = H_l + A_l/(G_{c,l} - G_l A_l)$ at discrete frequencies with $\sigma_v^2 = 0$, we have

$$\frac{\partial \bar{W}_l(n+1)}{\partial \bar{W}_l(n)} = 1$$
$$-\mu \frac{G_l^* P_l G_l}{\left|1 + G_l^*(\bar{W}_l^*(n) - H_l^*)\right|^2} \frac{1}{1 + G_l(\bar{W}_l(n) - H_l)}. \quad (33)$$

But from (32)

$$\frac{1}{1 + G(z)(W_{\text{opt}}(z) - H(z))} = 1 - z^{-q}A(z) \quad (34)$$

so that the stability condition is again given by (23).

4. EXAMPLES AND SIMULATION RESULTS

Here we assume that $d(n)$ is an m-th order AR (autoregressive) process with the innovation variance 1. That is, $R(z)$ is given by $R(z) = (1 - a_1 z^{-1} - \cdots - a_m z^{-m})^{-1}$ For $m = 1$, $A(z) = a_1^q$ where $|a_1| < 1$. Hence (23) is satisfied. For $m = 2$ and $q = 1$, $A(z) = a_1 + a_2 z^{-1}$ and (23) becomes $\text{Re}[1 - a_1 z^{-1} - a_2 z^{-2}] > 0$. For $m = 2$ and $q = 2$,$A(z) = a_1^2 + a_2 + a_1 a_2 z^{-1}$ and (23) becomes $\text{Re}[(1 + a_1 z^{-1})(1 - a_1 z^{-1} - a_2 z^{-2})] > 0$. For the latter problem $(1 + a_1 z^{-1})(1 - a_1 z^{-1} - a_2 z^{-2})$ must be a stable polynomial. Numerical investigations show that both regions coincide. In Fig.4 the learning curve shows the empirical variance of the squared error $e^2(n)$ of the former problem for the case of $m = 1, a_1 = 0.9, S(z) = z^{-1}$ and initial $\hat{S}(z) = 2.5z^{-1}$ with $N_w = 8, N_s = 2, \sigma_v^2 = 0.01, \mu_w = 0.0001, \mu_s = 0.01$. The variance is obtained by averaging over 50 data sets. In the early stage some instability is seen but the steady state variance is 1.0296 which is close to the lower bound 1.01. For the case of $m = 2, q = 2$, we see divergence if we set the parameters outside the above stability region. In Table 1, the steady state first 4 weights of the bias for the latter problem are presented together with the theoretical ones in (32) for the case of $m = 1, q = 1, a_1 = 0.9, H(z) = z^{-1}/10, G(z) = 4z^{-1}, N_w = 16, \mu = 0.0001$. The agreements are good.

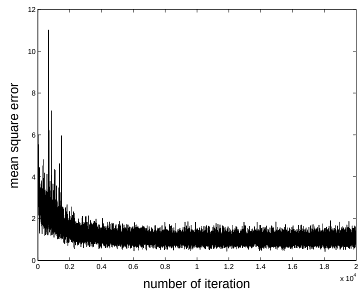

Fig. 4. Learning curve of the scheme in Fig.2.

	b_0	b_1	b_2	b_3
empirical	0.2232	0.1986	0.1790	0.1617
theoretical	0.2250	0.2025	0.1823	0.1640

Table 1. Theoretical and estimated bias in the hearing aid.

5. CONCLUSION

We have presented the analysis of the adaptive filter algorithms for the feedback-type ANC with on-line modeling of the feedback path and hearing aids concerning their stationary points and the local convergence conditions. The obtained theoretical results coincide well with the simulation results. A further study is needed to find schemes that do not require the condition (23).

6. REFERENCES

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