DESIGN OF NARROWBAND FIR FILTERS WITH MINIMAL NOISE GAIN USING COMPLEX INTERPOLATION

Konsta Koppinen

konsta.koppinen@tut.fi
Institute of Signal Processing
Tampere University of Technology
P.O.Box 553, FIN-33101 Tampere, Finland

ABSTRACT

It is shown that polynomial prediction is equivalent to requiring that the transfer function of the predictor interpolates the 'prediction-function' $f(z) = z$ and its derivatives at $z = 1$. This result is generalized to all other linear filtering operations, including interpolation, differentiation and smoothing, and to all other narrowband signal models, i.e. polynomially modulated complex exponential signals. An algorithm for determining the coefficients of this type of FIR filters with minimum noise gain is presented and illustrated by deriving the coefficients of predictive differentiators.

1. INTRODUCTION

Polynomial predictors are filters that extrapolate a polynomial signal model while attenuating wideband noise [1]. They are useful in situations where minimal delay or prediction of a smooth signal model is required, as in e.g. elevator control [2], DC-level detection in EEG [3] and zero-crossing detection in the synchronization of thyristor power converters [4].

Polynomials are a special class of signals in many ways, but one key property of polynomials which is shared by a much larger class of signals is that they are linearly predictable. This larger signal class of linearly predictable signals, which we also refer to as narrowband signals, is composed of arbitrary finite sums of polynomially modulated complex exponential signals.

In this paper we formulate the design problem of polynomial predictors in the frequency domain, and find that the constraint of polynomial prediction is equivalent to the requirement that the frequency response of the predictor interpolates the function $f(z) = z$ and its derivatives at $z = 1$. We can interpret this as follows: although polynomial signals strictly have no z-transform, they can be represented by line spectra and derivatives at $z = 1$, and the function $f(z) = z$ is the transfer function of the (causally unrealizable) ideal one-step–ahead predictor. By extending this analysis to other points besides $z = 1$ and other functions than $f(z) = z$ we see that the requirements for a filter to act like any given ideal filter for a class of narrowband signals corresponds to interpolation of the ideal transfer function at points in the complex plane specified by the narrowband signal model.

A design method for finding the optimal coefficients of this class of FIR filters with minimal noise gain has been presented in [5]. The design of predictive differentiators is illustrated as a special case.

2. POLYNOMIAL PREDICTION AS COMPLEX INTERPOLATION

Let $h(n)$ be the impulse response of a one-step–ahead FIR predictor of polynomial order L, i.e. we have

$$\sum_{k=0}^{N} h(k)(-k)^{\ell} = 1,\ \ell = 0, 1, \ldots, L, \tag{1}$$

where the length of the filter is $N + 1$. This condition guarantees that

$$\sum_{k=0}^{N} h(k)(n-k)^{\ell} = (n+1)^{\ell},\ \ell = 0, 1, \ldots, L,$$

for all n.

It has been shown [6] that $H_{L+1}(z)$ is the transfer function of a polynomial predictor of order $L + 1$ if and only if $H_L(z)$ is a predictor of order L. Based on this result we can prove the following:

Lemma. If $H(z)$ is a polynomial predictor of order L, then

$$\begin{aligned} H(z)|_{z=1} &= 1, \\ H^{(1)}(z)|_{z=1} &= 1,\ \text{if } L \geq 1, \\ H^{(\ell)}(z)|_{z=1} &= 0,\ \ell = 2, 3, \ldots, L, \end{aligned}$$

where $H^{(\ell)}(z)|_{z=z_0}$ denotes the ℓ^{th} derivative of $H(z)$ evaluated at $z = z_0$.

Proof. For $L = 0$, based on eq. (1), we have

$$H(z)|_{z=1} = \sum_{k=0}^{N} h(k)z^{-k}\big|_{z=1} = \sum_{k=0}^{N} h(k) = 1, \tag{2}$$

and similarly for $L = 1$,

$$\begin{aligned} H^{(1)}(z)|_{z=1} &= \sum_{k=0}^{N} h(k)(-k)z^{-k-1}\big|_{z=1} &(3)\\ &= \sum_{k=0}^{N} h(k)(-k) &(4)\\ &= 1. &(5) \end{aligned}$$

Assume now that the result holds for predictors of order at most L and consider a predictor $H_{L+1}(z)$ of order $L+1$, where $L \geq 1$. Based on the result mentioned above, we can express $H_{L+1}(z)$ as

$$H_{L+1}(z) = 1 + (1 - z^{-1})H_L(z),$$

where $H_L(z)$ is a predictor of order L. Now calculating the $(L+1)^{\text{th}}$ derivative of $H_{L+1}(z)$ at $z = 1$ using Leibnitz's rule [7]

$$\left(\frac{d}{dz}\right)^{L} f(z)g(z) = \sum_{\ell=0}^{L} \binom{L}{\ell} f^{(L-\ell)}(z)g^{(\ell)}(z),$$

we get

$$\begin{aligned} &\left(\frac{d}{dz}\right)^{L+1} (1 + (1 - z^{-1})\, H_L(z))|_{z=1} \\ =\ &\left(\frac{d}{dz}\right)^{L+1} (1 - z^{-1})\, H_L(z)|_{z=1} \\ =\ &\sum_{\ell=0}^{L+1} \binom{L+1}{\ell} \left(\frac{d}{dz}\right)^{L+1-\ell} (1 - z^{-1})\, H_L^{(\ell)}(z)\Big|_{z=1} \\ =\ &\binom{L+1}{0}(-1)^{L+2}(L+1)! + \binom{L+1}{1}(-1)^{L+1}L! \\ =\ &(-1)^{L+2}(L+1)! + (L+1)(-1)^{L+1}L! \\ =\ &0, \end{aligned}$$

where we've used eqs. (2), (5), the fact that

$$\left(\frac{d}{dz}\right)^{\ell} (1 - z^{-1})\big|_{z=1} = (-1)^{\ell+1}\ell!$$

and the induction hypothesis $H^{(\ell)}(z)|_{z=1} = 0$ for $\ell = 2, 3, \ldots, L$.

There is a good explanation for the values of the derivatives of polynomial predictors given by the previous lemma: the sequence $1, 1, 0, 0, \ldots$ is composed of the values of the derivatives of the function $f(z) = z$ at $z = 1$. The function $f(z) = z$ in turn is the transfer function of the ideal one-step–ahead predictor. Thus, we get the following interpretation: polynomial predictors of order L are exactly those filters which have a transfer function which interpolates $f(z) = z$ and its first L derivatives at $z = 1$.

This raises the question of what happens if we interpolate *other* transfer functions, for example the ideal *two*-step–ahead predictor function $f(z) = z^2$ and its derivatives at $z = 1$? Does this give us a two-step–ahead polynomial predictor? It turns out, in fact, that it does, as we will show in the next section.

Furthermore, by the choice of the points in the complex plane which interpolate the ideal transfer function we can select the set of signals for which the interpolated transfer function behaves as the ideal filter. For example, if we interpolate the desired function at $z = \exp(\pm j\omega_0)$, the filter behaves like the ideal filter for the signals $\sin(\omega_0 n + \phi)$, where ϕ is any real number [8].

3. FILTERING AS COMPLEX INTERPOLATION

Let $I(z)$ be the ideal transfer function we wish to approximate, and choose points $z_0, z_1, \ldots, z_K$ in the complex plane with corresponding multiplicities $M_0, M_1, \ldots, M_K$. Then we have the following result.

Theorem. Let $H(z)$ be the transfer function of any causal FIR filter that interpolates $I(z)$ and its first M_k derivatives at the point $z = z_k$, i.e. we have

$$H^{(m)}(z_k) = I^{(m)}(z_k),\ k = 0, 1, \ldots, K,\ m = 0, 1, \ldots, M_k.$$

Then the output of the filter $H(z)$ for the input signal

$$x(n) = n^L z_k^n,$$

where $L \leq M_k$, is equal to the output of the filter $I(z)$ to the input $x(n)$.

Proof. Consider the signal $x(n) = n^L z_0^n$. We will show that the output $y(n)$ of the filter $H(z)$ to the input $x(n)$ is determined completely by the values of the first L derivatives of $H(z)$ at $z = z_0$. We have

$$\begin{aligned} y(n) &= \sum_{k=0}^{N} h(k)(n-k)^L z_0^{n-k} \\ &= \sum_{k=0}^{N} h(k) \sum_{\ell=0}^{L} \binom{L}{\ell} n^{L-\ell}(-k)^{\ell} z_0^{n-k} \\ &= z_0^n \sum_{\ell=0}^{L} \binom{L}{\ell} n^{L-\ell} \sum_{k=0}^{N} h(k)(-k)^{\ell} z_0^{-k}, \end{aligned}$$

so the output depends only on the set of quantities

$$\left\{ \sum_{k=0}^{N} h(k)(-k)^{\ell} z_0^{-k} \right\}_{\ell=0}^{L}. \tag{6}$$

The value of the ℓ^{th} derivative of $H(z)$ at $z = z_0$ is

$$\begin{aligned}
&\left(\frac{d}{dz}\right)^{\ell} H(z)\big|_{z=z_0} \\
=\ & \left(\frac{d}{dz}\right)^{\ell} \sum_{k=0}^{N} h(k) z^{-k}\big|_{z=z_0} \\
=\ & \left(\frac{d}{dz}\right)^{\ell-1} \sum_{k=0}^{N} h(k)(-k)\ z^{-k-1}\big|_{z=z_0} \\
=\ & \cdots \\
=\ & \sum_{k=0}^{N} h(k)(-k)(-k-1)\cdots(-k-\ell+1)\ z^{-k-\ell}\big|_{z=z_0} \\
=\ & z_0^{-\ell} \sum_{k=0}^{N} h(k)(-k)(-k-1)\cdots(-k-\ell+1) z_0^{-k}.
\end{aligned}$$

However, we can express the set of values

$$\left\{ z_0^{-\ell} \sum_{k=0}^{N} h(k)(-k)\cdots(-k-\ell+1) z_0^{-k} \right\}_{\ell=0}^{L} \tag{7}$$

as a linear combination of the values in (6). This transformation is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & z_0^{-1} & 0 & 0 & \cdots \\ 0 & -z_0^{-2} & z_0^{-2} & 0 & \cdots \\ 0 & 2z_0^{-3} & -3z_0^{-3} & z_0^{-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This matrix is lower diagonal, and thus its inverse exists and can be used to write (6) in terms of (7).

The same reasoning applies to the points $z_1, \ldots, z_K$ as well, showing that the output of $H(z)$ for the signals

$$n^m z_k^n,\ k = 0, 1, \ldots, K,\ m = 0, 1, \ldots, M_k \tag{8}$$

depends only on the derivatives of $H(z)$ of order M_0, M_1, …, M_K at $z_0, \ldots, z_K$. But because we require that $H(z)$ and its derivatives match $I(z)$ and its derivatives at these points with the appropriate orders, it follows that the output of $H(z)$ matches the output of $I(z)$ for the signals given by (8), completing the proof.

Note that the same result applies to *any* function $I(z)$ and as is well known, there are several interesting functions around that define useful filters, for example:

Prediction/smoothing/interpolation. The function $I(z) = z^p$ is the transfer function of the ideal p-step–ahead predictor, if p is a positive integer. If we let $p = 0$ we obtain a smoothing filter that gives an unbiased estimate of the current sample value with reduced wideband noise, if the filter is designed to have minimal noise gain. If p is not an integer, $I(z) = z^p$ is the transfer function of the ideal predictor and interpolator.

Differentiation. The ideal differentiator produces the output $(j\omega)\exp(j\omega n)$ to the input $\exp(j\omega n)$, from which it follows that the transfer function of the ideal differentiator is $I(z) = \log(z)$. By cascading differentiators, we find that the ideal transfer function of the filter for calculating the n^{th}-order derivative is $I(z) = (\log(z))^n$.

Lowpass/bandpass/highpass filtering. Here the ideal transfer function is 1 for z's with angles in the passband of the filter, and 0 elsewhere.

We can also cascade these transfer functions to yield combinations of the above properties. For example, we can cascade a differentiator with a polynomial predictor to give a polynomial predictive differentiator, as shown in the next section.

4. DESIGN OF NARROWBAND FILTERS WITH MINIMAL NOISE GAIN

Consider a vector of FIR filter coefficients $\mathbf{h}$ which satisfy a set of linear constraints:

$$\mathbf{S}^T \mathbf{h} = \mathbf{p},$$

where $\mathbf{S}$ is a $N \times K$-matrix with $K \leq N$. If we want the filter to have minimal noise gain, i.e. to minimize the impulse response energy with the given constraints, the filter coefficients are given by the minimum-norm solution [5]

$$\mathbf{h} = \mathbf{S}(\mathbf{S}^T\mathbf{S})^{-1}\mathbf{p}. \tag{9}$$

We illustrate the choice of $\mathbf{S}$ and $\mathbf{h}$ by designing a two-step–ahead predictive differentiator for 2^{nd}-degree polynomials and damped sinusoidals $0.9^n \sin(0.1\pi n)$ with arbitrary phase. The same method can be used to design a filter with other ideal transfer functions and signal models.

The ideal transfer function for the given signal model is the cascade of the ideal two-step predictor and the ideal differentiator, i.e.

$$I(z) = z^2 \log(z).$$

The first 2 derivatives of $I(z)$ are $I^{(1)}(z) = 2z\log(z) + z$ and $I^{(2)}(z) = 2\log(z) + 3$. Thus, the interpolation requirements are: $I(1) = 0$, $I^{(1)}(1) = 1$, $I^{(2)}(1) = 3$, $I(z_0) = z_0^2 \log(z_0) = -0.219 + 0.156j$, $I(z_0^*) = (z_0^*)^2 \log(z_0^*) = -0.219 - 0.156j$, where $z_0 = 0.9\exp(j0.1\pi)$ and $()^*$ denotes complex conjugation.

Now for a given filter length N we set

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & z_0^0 & (z_0^*)^0 \\ 1 & -1 & (-1)^2 & z_0^{-1} & (z_0^*)^{-1} \\ 1 & -2 & (-2)^2 & z_0^{-2} & (z_0^*)^{-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -N+1 & (-N+1)^2 & z_0^{-N+1} & (z_0^*)^{-N+1} \end{bmatrix}$$

and

$$\mathbf{p} = \begin{bmatrix} 0 & 1 & 3 & -0.219+0.156j & -0.219-0.156j \end{bmatrix}^T,$$

and calculate the optimal filter coefficients using eq. (9).

The impulse and magnitude responses of the optimal predictor of length 50 are shown in Fig.1. The noise gain of this filter is 0.0021985. An example of filtering a noisy signal conforming to the signal model with this filter is shown in Fig. 2, where the output of the optimal filter of length 10 (noise gain 13.662) is shown for comparison.

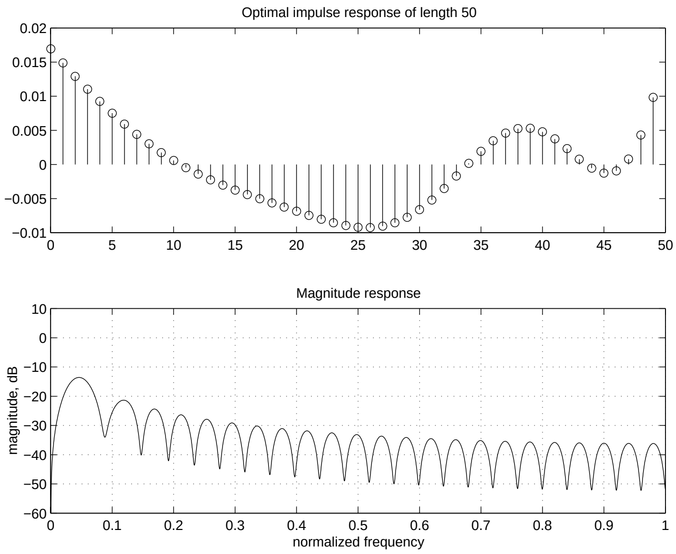

Fig. 1. Optimal predictive differentiator of length 50.

5. CONCLUSIONS

In this paper we showed that polynomial prediction can be interpreted as interpolating the 'prediction-function' $f(z) = z$ and its derivatives at $z = 1$. By extending this reasoning, we showed that *any* filtering operation, including prediction, smoothing and interpolation, for a narrowband signal model is equivalent to interpolating the ideal transfer function at the points in the complex plane defined by the signal model. We illustrated this approach by deriving the coefficients of the optimal FIR filters for predictive differentiation of a sum of polynomials and sinusoids.

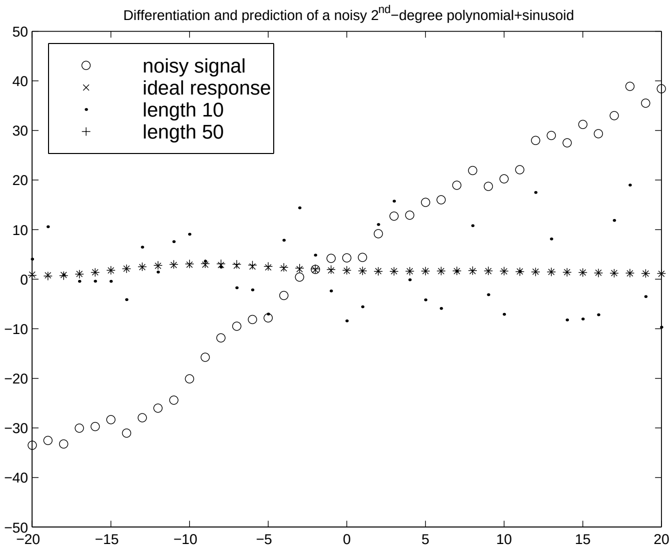

Fig. 2. Predictive differentiation of a polynomial+sinusoidal signal with additive white noise.

6. REFERENCES

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