RECOVERY OF DIGITIZED SIGNALS USING SLEPIAN FUNCTIONS

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ABSTRACT

The continuous prolate spheroidal wave functions (Slepian functions) were found to be useful for analog signal processing several decades ago. But the digital revolution left them in the dust since they did not seem naturally adapted to discrete analysis. Yet they have many desirable, even unique, properties that originally made them fascinating and could lead to some applications in digital and analog-digital signal processing practice. The simplest such applications involve the conversion of a digital signal to an analog signal and recovery of a band-limited signal from their values at countable many distinct points on the real line. The Shannon sampling theorem, which is based on the sinc expansions, plays the most important theoretical foundation in such approaches. In this work, by using the natural connection between the Slepian functions and sinc function, several new formulae based on integer values of Slepian functions are developed. These formulae are then used to replace the sinc function in sampling theorems for digitizing band-limited signals. Finally, they are used to construct analysis and synthesis filter banks for sampled values of a band-limited signal.

1. INTRODUCTION

The continuous prolate spheroidal wave functions (Slepian functions), because of a "lucky accident" [2], were found to be quite useful for analog signal processing several decades ago. But the digital revolution left them in the dust since they did not seem naturally adapted to discrete analysis. Yet they have many desirable, even unique, properties that originally made them useful and could lead to useful new digital and analog-digital techniques. The simplest such involves the conversion of a digital signal to an analog.

One way of doing this is by means of the Shannon sampling theorem [1] given by the formula

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f(nT)\frac{\sin\sigma(t-nT)}{\sigma(t-nT)} \qquad (1) \\ &= \sum_{n=-\infty}^{\infty} f(nT)S((t/T-n) \end{aligned}$$

where $T=\pi/\sigma$, which is valid for $\sigma-$ bandlimited function with finite energy, that is, for continuous functions in $L_2(R)$ whose Fourier transform has support in $[-\sigma,\sigma]$. These are signals which are invariant under a perfect filter with frequency no greater than σ. This theorem has become a well-known part of both the mathematical and engineering literature (see the recent book by Higgins [1] or the paper by Vaidyanathan [3]). It also falls naturally into a "wavelet" setting since the sinc function appearing in (1.1), $S(t)=\sin\pi t/\pi t$, is a particular example of a "scaling function" appearing in wavelet theory [4]. This sinc function is also closely related to the Slepian functions $\{\varphi_n\}$ (sometimes called Slepian functions) which themselves have some interesting sampling properties which will be the chief subject of this work.

The $\{\varphi_n\}$ constitute an orthonormal basis of the space of $\sigma-$bandlimited functions on the real line just as the translates of $S(t)$, $\{S(t/T-n)\}$ do. There are several ways of characterizing them; they can be defined by the integral equation

$$\int_{-\tau}^{\tau}\varphi_n(x)\frac{1}{T}S(\frac{t-x}{T})dx=\lambda_n\varphi_n(t), \qquad (1.2)$$

or by the differential equation

$$(\tau^2-t^2)\frac{d^2\varphi_n}{dt^2}-2t\frac{d\varphi_n}{dt}-\sigma^2t^2\varphi_n=\mu_n\varphi_n, \qquad (1.3)$$

or as the maximum energy concentration of a $\sigma-$bandlimited function on the interval (-τ,τ); that is φ_0 is the function of total energy 1 such that

$$\int_{-\tau}^{\tau}|f(t)|^2dt$$

is maximized, φ_1 is the function with the maximum energy concentration among those functions orthogonal to φ_0, etc.

2. SOME PROPERTIES OF SLEPAIN FUNCTIONS.

The $\{\varphi_n\}$ discussed above depend on a parameter τ which comes from the interval of concentration and on σ since they are $\sigma-$ bandlimited functions. We shall usually suppress these parameters, but sometimes when needed, we shall use the notation $\varphi_n = \varphi_{n,\sigma,\tau}$.

In addition to the equation (1.2), the $\{\varphi_n\}$ satisfy an integral equation over $(-\infty, \infty)$ as well:

$$\int_{-\infty}^{\infty} \varphi_n(x)\frac{1}{T}S(\frac{t-x}{T})dx = \varphi_n(t). \qquad (2.1)$$

This leads to a dual orthogonality

$$\begin{aligned} \int_{-\tau}^{\tau} \varphi_n(x)\varphi_m(x)dx &= \lambda_n\delta_{nm}, \qquad (2) \\ \int_{-\infty}^{\infty} \varphi_n(x)\varphi_m(x)dx &= \delta_{nm}. \end{aligned}$$

In fact, they constitute an orthogonal basis of $L^2(-\tau,\tau)$, and an orthornormal basis of the subspace B_σ of L^2(-$\infty,\infty), the Paley-Wiener space of all \sigma-$bandlimited functions.

These equations lead to still another type of orthogonality for the Slepian functions, two discrete orthogonality conditions of the form

$$\begin{aligned} T\sum_{n=0}^{\infty} \varphi_n(kT)\varphi_n(mT) &= \delta_{mk}, \qquad (3) \\ T\sum_{k=-\infty}^{\infty} \varphi_n(kT)\varphi_m(kT) &= \delta_{mn}. \end{aligned}$$

The first result may be used to find the expansion coefficients for other $\sigma-$bandlimited functions by using only the sampled values at the integers.

Proposition 1 Let $f \in B_\sigma$, let $\{\varphi_n\}$ denote a sequence of $\sigma-$bandlimited Slepian functions with any concentration interval $(-\tau,\tau)$ with $\tau > 0$; let $T = \pi/\sigma$; then

$$f(t) = \sum_{n=0}^{\infty}\{\sum_{k=-\infty}^{\infty} T\varphi_n(kT)f(kT)\}\varphi_n(t). \qquad (2.4)$$

The orthogonality on the interval (-τ,τ) can be used to get another set of formulae. One such is Parseval's equality for a function $f \in B_\sigma$ restricted to this interval

$$\int_{-\tau}^{\tau} |f(x)|^2dx = \sum_{n=0}^{\infty}\lambda_n|a_n|^2.$$

This latter series may be well approximated by a finite sum because of the unique behavior of the eigenvalues $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq ... > 0$. The first $[\sigma\tau]$ are relatively close to 1 while the remaining ones are close to 0 [2] from which it follows that

$$\int_{-\tau}^{\tau} |f(x)|^2dx \approx \sum_{n=0}^{[\sigma\tau]}\lambda_n|a_n|^2 \qquad (2.5)$$

to a high degree of approximation.

3. A SAMPLING THEOREM BASED ON SLEPIAN FUNCTIONS

We consider first a $\pi-$bandlimited signal $f(t)$ (i.e. $\sigma = \pi$) for simplicity whose energy is mainly concentrated on (-τ,τ). It cannot have a higher proportion of its energy on this interval than does φ_0 of course. But if we try to use the standard sampling theorem (1.1), we see that the sampling approximation does not reflect this concentration since the partial sums have a high proportion of their energy outside of the interval. This happens since we have the following inequality for the sinc function

$$\int_{\tau}^{\infty} |S(x)|^2dx \geq \frac{1}{\pi^2[\tau+1]}.$$

Thus it appears that the use of the sinc function for sampling such a signal is not very efficient.

Table 1 below shows the eigenvalues (concentration index) for π bandlimited Slepian functions with time concentration interval $(-\tau,\tau)$, $\tau = 1,2$. Note that for $\tau = 1$,the eigenvalues are negligible for $k > 4$, while for $\tau = 2$, they are negligible for $k > 7$, and hence the sum in (2.5) can be terminated after 5 and 8 terms respectively for the two cases.

Table 1. Concentration index of Slepian functions

k	$\lambda(1,k)$	$\lambda(2,k)$
0	9.810463e-001	9.999428e-001
1	7.496202e-001	9.975617e-001
2	2.435930e-001	9.593904e-001
3	2.464655e-002	7.217516e-001
4	1.066060e-003	2.746661e-001
5	2.741514e-005	4.301465e-002
6	4.802783e-007	3.478239e-003
7	6.133203e-009	1.870286e-004
8	5.971410e-011	7.465624e-006
9	4.582026e-013	2.324184e-007
10	2.843140e-015	5.820375e-009

If we are interested in the behavior of our signal in the interval $(-\tau, \tau)$ only, then we have for $\pi-$bandlimited $f(t)$, a truncated version of the sampling formula (2.4)

$$f(t) = \sum_{k=-\infty}^{\infty} f(k)\{\sum_{n=0}^{N} \varphi_n(k)\varphi_n(t)\} \qquad (3.1)$$

It holds within a degree of accuracy needed for N sufficiently large.

While individual values of $f(k)$ may be large, they cannot be too large since they are Fourier coefficients of $\widehat{f}(\omega)$, and if the latter is sufficiently smooth, must converge to zero rapidly. Thus it is plausible that $f(t)$ may be represented by the truncated series:

$$f(t) \approx f_\tau(t) = \sum_{k=-\tau}^{\tau} f(k)\{\sum_{n=0}^{[\pi\tau]} \varphi_n(k)\varphi_n(t)\}\chi_\tau(t). \qquad (3.2)$$

The error in using the truncation in (3.2) in the mean square sense may now be found when $\widehat{f}$ is sufficiently smooth (i.e. belongs to the Sobolev space H^α for $\alpha \geq p$, where p is a positive integer). It can be shown to satisfy

$$\int_{-\tau}^{\tau} |f(t) - f_\tau(t)|^2 dt \leq C\tau^{2-2p}$$

for some constant C.

3.1. Discrete sequences

The previous discussion gives us a sampling theorem suitable for analog signals, i.e., functions of a real variable t. However, most of the modern applications involve discrete sequences and their processing. Frequently, one has only the sampled values of the signal which, in the case of $\pi-$ bandlimited signals, are given by the series of Slepian functions

$$f(k) = \sum_n a_n\varphi_n(k).$$

The inverse is given by the series

$$a_n = \sum_k f(k)\varphi_n(k)$$

because of the orthogonality given by (2.3) which allow us to find either series from the other.

Again, the two series may be truncated for signals concentrated on $[-\tau, \tau]$ so that the first becomes

$$f(k) = \sum_{n=0}^{[\pi\tau]} a_n\varphi_n(k) \qquad (3.3)$$

and the second becomes approximately

$$a_n = \sum_{|k|<\tau} f(k)\varphi_n(k). \qquad (3.4)$$

This allows us to interpret them as filter banks in which (33) corresponds to the analysis filter bank, while (3.4) gives the synthesis filter bank, with the concatenation of the two giving (almost) perfect reconstruction.

These formulas can also be used in conjunction with (3.2) to give interpolation much as has been done for other sampling theorems [3]. We can use (3.2) to obtain

$$f(k/L) = \sum_n a_n\varphi_n(k/L) \qquad (3.5)$$

for a finer sampling spacing $1/L$. Thus we can proceed by beginning with a sequence $\{f(k)\}$ and use (3.4) to find the sequence $\{a_n\}$ and then (3.5) for the interpolated values.

4. MRA BASED ON SLEPIAN FUNCTIONS

Because of the discrete orthogonality, the series formula for a $\sigma-$bandlimited function may be given by

$$f(t) = \sum_{n=0}^{\infty} b_n\varphi_n(t), \; b_n = \sum_k Tf(kT)\varphi_n(kT)$$

for Slepian functions concentrated on $(-\tau, \tau)$. The relation between the expansions at various frequency or time scales may be found by using the integral equation expression (2.1). By a straightforward change of scale in the integral equation, we find that

$$\varphi_{n,\sigma\tau,1}(x) = \varphi_{n,\sigma,\tau}(\tau x).$$

We can also use the series expansions to get a relation between, say, π and $2\pi-$ bandlimited Slepian functions since $\varphi_{n,\pi,\tau} \in B_{2\pi}$. We can express the former as

$$\varphi_{n,\pi,\tau}(t) = \sum_{k=0}^{\infty} c_{nk}\varphi_{k,2\pi,\tau}(t)) \qquad (4.1)$$

where

$$c_{nk} = \frac{1}{2}\sum_m \varphi_{n,\pi,\tau}(m/2)\varphi_{k,2\pi,\tau}(m/2). \qquad (4.2)$$

This may be considered as one form of a dilation equation, which relates the various subspaces in a multiresolution analysis (MRA) which appears in wavelet theory. In fact the Paley-Wiener spaces constitute such an MRA $\{V_m\}$, where $V_m = B_{2^m\pi}$, the space of $2^m\pi-$ bandlimited functions.

Proposition 2 let $f \in V_0$ with Slepian functions series in V_0 given by

$$f(t) = \sum_{k=0}^{\infty} a_k^0 \varphi_{k,\pi,\tau}(t)$$

and with Slepian functions series in V_1 given by

$$f(t) = \sum_{k=0}^{\infty} a_k^1 \varphi_{k,2\pi,\tau}(t);$$

then the coefficients are related by

$$a_n^0 = \sum_{k=0}^{\infty} c_{nk} a_k^1 \qquad (4.3)$$

where the c_{nk} are given by (4.2).

The formulae in this proposition extend to other scales as well, but they only work if our function belongs to the space V_m at the coarsest scale. If we wish to emulate the decomposition and reconstruction algorithms of wavelet theory [5], we need to find an associated basis of the orthogonal complement of V_0 in V_1. This is an interesting but not too difficult undertaking and will be touched on in a separate work [6]. As in the case of wavelets, we can find the rate of convergence of either the sampled series or the standard orthogonal series in V_m to a function with the required of smoothness. This is also taken up in [6] where a number of numerical examples illustrating the theory are found.

References

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