LINEAR PHASE LAGRANGE INTERPOLATION FILTER USING ODD NUMBER OF BASEPOINTS

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ABSTRACT

Interpolation filters are used to calculate new samples at arbitrary time instants in between existing discrete-time samples. Polynomial-based interpolation filters can be efficiently implemented using Farrow structure. Lagrange coefficients are often used to describe such classical polynomial interpolators. Previous references have concluded that there must be an even number of samples in the basepoint set to perform interpolation in order to satisfy linear phase requirement. This paper introduces a new method to construct linear phase Lagrange interpolator using an odd number of basepoints. Although the conceptual analog reconstruction filter does not have a time-continuous impulse response, it can be proved that the interpolation results are time-continuous within the approximation error of polynomial-based interpolation.

1. INTRODUCTION

Interpolation filters are used to calculate new samples at arbitrary time instants in between existing discrete-time samples. They are required whenever there is a need to change from one sampling rate to another [1], e.g., the digital interfaces between different digital audio equipment [2], or to change from one sampling instant to another, e.g., the symbol synchronizations in digital receivers [3] and [4].

One possible implementation of such interpolation filters is to use traditional FIR filter structures to calculate $x[(n+\mu)T_s]$ from series of input samples $x[nT_s]$; $\mu \in [0,1)$ and is often referred as fractional interval. For a limited number of $\{\mu_k\}$, the filter coefficients can be pre-computed and stored in the memory for each individual μ_k. When the number of $\{\mu_k\}$ becomes very large to meet fine resolution requirement of interpolation interval, the size of the memory can become very large thus making this technique more expensive to implement.

Classical polynomial interpolation methods provide another way to implement such interpolation filters. Polynomial-based filters do not have the optimum filter response, but they can eliminate the need to store pre-computed coefficients since they can be computed on-line based on the interpolation interval μ_k. In addition, a special FIR structure, known as Farrow structure and permits simple handling of the coefficients, is only applicable for polynomial based filters [4].

Previous references concluded that in order to construct a linear phase interpolation filter, there must be an even number of samples in the basepoint set [1] and [4]. The advantage of using linear phase interpolation filter is to avoid delay distortion. In this paper, we introduce a method to construct linear phase Lagrange interpolation filter using odd number of samples in the basepoint set. Although the impulse response of the conceptual analog reconstruction filter is not time-continuous, it can be proved that the interpolation results are time-continuous within the approximation error of polynomial-based interpolation.

2. INTERPOLATIONS AND LAGRANGE POLYNOMIAL-BASED INTERPOLATIONS

In many applications of digital communications and digital signal processing (DSP), what is known about the signal is the frequency band of interest rather than its deterministic function with respect to time. It is often preferred to analyze the frequency domain behavior of the interpolation filter, in addition to the time domain behavior. Because interpolation is essentially a reconstruction problem, the interpolation filters can be analyzed using the hybrid analog/digital conceptual model shown in Fig. 1 [5]. In this model, the interpolated outputs are obtained by sampling the reconstructed analog signal $y_a(t)$ at the time instants $t=kT_{out}=(n_k+\mu_k)T_{in}$, where T_{in} and T_{out} are the input and output sampling intervals respectively. This model is only a conceptual model since almost all interpolators are implemented in an all-digital fashion.

One of the frequency domain behaviors of interpolation is to remove the images of the signal spectrum that are located at integer multiples of $2\pi/T_{in}$. The underlying analog filter with impulse response $h_a(t)$ is the prototype lowpass filter (LPF) that can be used to evaluate such behavior. The sampled form of this analog filter can be used to calculate the new interpolated samples entirely digitally [3].

Polynomial-based interpolation is a classical numerical technique to fit the original Q samples with a polynomial having an order of Q-1. Lagrange coefficients are often used to describe such polynomials. The impulse response of the analog conceptual reconstruction filter can be constructed by applying Lagrange coefficient formulas in each interval between basepoints (original sampling instants), thus the filter response is a piecewise polynomial of μ [1]. The impulse response $h_a(t)$ is non-zero only in the intervals limited by the Q original sampling points. Fig. 2 shows the construction of such piecewise polynomial based on Lagrange coefficients $A_k^Q(\mu)$, where μ is the aforementioned fractional interval. The Lagrange coefficients $A_k^Q(\mu)$ are given by the equations [1]

$$A_k^Q(\mu) = \frac{(-1)^{k+Q/2}}{(\frac{Q-2}{2}+k)!(\frac{Q}{2}-k)!(\mu-k)} \bullet \prod_{i=1}^{Q}(\mu+\frac{Q}{2}-i), \quad \text{Q even} \quad (1a)$$

$$A_k^Q(\mu) = \frac{(-1)^{k+(Q-1)/2}}{(\frac{Q-1}{2}+k)!(\frac{Q-1}{2}-k)!(\mu-k)} \bullet \prod_{i=0}^{Q-1}(\mu+\frac{Q-1}{2}-i), \quad \text{Q odd} \quad (1b)$$

Fig. 3 shows an impulse response of a cubic interpolator using (1a) and the process described in Fig. 2.

3. LINEAR PHASE LAGRANGE INTERPOLATOR USING ODD NUMBER OF BASEPOINTS

What is shown in Fig. 3 is a polynomial-based interpolator using an even number (Q=4) of basepoints, which implies a polynomial of odd degree (Q-1). It is obvious to see such filter has an even symmetrical impulse response about t=0, i.e., $h_a(t) = h_a(-t)$. Schafer and Rabiner [1] and [4] showed that in order to obtain a unique basepoint set for an interpolant and avoid delay distortion as well: 1) there must be an even number of samples in the basepoint set and 2) interpolation should be performed only in the central interval of the basepoint set. They [1] also concluded that when the number of original samples in the basepoint set (Q) is odd, the impulse response of Lagrange interpolation does not have linear phase if the filter is constructed using the process described in Fig. 2.

An intuitive explanation is that there are an odd number of intervals between the basepoints when using an even number of basepoints, so there is a central interval. If an odd number of basepoints are selected, no central interval exists since the number of intervals between the basepoints is even.

But there is a central point when an odd number of basepoints are employed, an intuitive alternative to divide the interpolation interval is: instead of dividing the interpolation intervals from $(n-1)T_s$ to nT_s, where n is any integer and T_s is the original sampling interval, divide them from $(n-1/2)T_s$ to $(n+1/2)T_s$. With this method, a central interval is constructed spanning from half sampling interval before central point to half sampling interval after it. Fig. 4 shows the construction of such piecewise polynomial using proposed interpolation intervals when Q is odd. Equation (1b) will still be used as the formula to calculate Lagrange coefficients. The difference is when Q is odd, the basepoint index [3] should be derived by $\text{round}[kT_{out}/T_{in}]$ rather than $\text{int}[kT_{out}/T_{in}]$, where round[z] means the common rounding operation of z and int[z] means the largest integer not exceeding z [3]. In addition, the described method suggests the fractional interval belongs to [-0.5, 0.5) instead of [0, 1) as used in many references [3], [4] and [6].

From Fig. 4, we can see that in order for such a filter to have an even symmetrical impulse response about t=0, the piecewise polynomial given by (1b) must satisfy:

$$A_k^Q(\mu) = A_{-k}^Q(-\mu) \quad (2)$$

From (1b), it can be derived that when μ≠0,

$$\frac{A_k^Q(\mu)}{A_{-k}^Q(-\mu)} = (-1)^{2k} \frac{\dfrac{\mu\prod_{i=0}^{\frac{Q-3}{2}}[\mu^2-(\frac{Q-1}{2}-i)^2]}{(\mu-k)(\frac{Q-1}{2}+k)!(\frac{Q-1}{2}-k)!}}{\dfrac{(-\mu)\prod_{i=0}^{\frac{Q-3}{2}}[\mu^2-(\frac{Q-1}{2}-i)^2]}{-(\mu-k)(\frac{Q-1}{2}-k)!(\frac{Q-1}{2}+k)!}} = 1 \quad (3)$$

From (3), the impulse response of the reconstruction filter is symmetrical about t=0. Therefore, a linear phase interpolation filter based on proposed Lagrange piecewise polynomial with odd number of samples in the basepoint set can be constructed. Fig. 5 shows an impulse response of a 4th order interpolation filter using (1b) and the process described in Fig. 4.

4. CONTINUITY OF INTERPOLATION RESULTS USING PROPOSED FILTER

The impulse response $h_a(t)$ constructed by proposed Lagrange piecewise polynomial shown in Fig. 5 is not a continuous function of t. It can be seen from (1b) that

$$A_k^Q(0.5) \neq A_{k-1}^Q(-0.5), -\frac{Q-1}{2} < k \leq \frac{Q-1}{2} \quad (4)$$

In this section, Q is assumed to be odd unless specified otherwise.

As illustrated in previous section, the interpolation interval μ ranges between –0.5 and 0.5 when an odd number of basepoints are employed, so the discontinuity in $h_a(t)$ locates at the boundary of interpolation intervals shown in Fig. 4. Since interpolation is a time invariant process, it can be assumed the interpolation is performed in the interval between 0 and 1 without losing generality. In order to prove the continuity of the interpolation results, it needs to prove that $x(0.5^-) = x(0.5^+)$, where $x(0.5^-)$ is the left-hand limit of the interpolation results x(t) at t=0.5, and $x(0.5^+)$ is the right-hand limit.

As shown in Fig. 6, the calculation of $x(0.5^-)$ and $x(0.5^+)$ involves 2 different sets of basepoints, both consisting of Q original samples. As described in [1], these two interpolants can be evaluated by

$$x(0.5^-) = \sum_{k=-\frac{Q-1}{2}}^{\frac{Q-1}{2}} A_k^Q(0.5)x(k) \qquad (5a)$$

$$x(0.5^+) = \sum_{k=-\frac{Q-1}{2}}^{\frac{Q-1}{2}} A_k^Q(-0.5)x(k+1) \qquad (5b)$$

Subtract (5b) from (5a)

$$x(0.5^-) - x(0.5^+) = A^Q_{-\frac{Q-1}{2}}(0.5)x(-\frac{Q-1}{2}) + \sum_{k=-\frac{Q-3}{2}}^{\frac{Q-1}{2}} D_k^Q x(k) - A^Q_{\frac{Q-1}{2}}(-0.5)x(\frac{Q+1}{2}) \qquad (6)$$

where $D_k^Q=A_k^Q(0.5)-A_{k-1}^Q(-0.5)$, $k\in[-(Q-3)/2, (Q-1)/2]$.

According to (1b) and (3),

$$A^Q_{-\frac{Q-1}{2}}(0.5) = A^Q_{\frac{Q-1}{2}}(-0.5) = \frac{2C}{Q!} \qquad (7)$$

where $C = \prod_{i=0}^{Q-1}(0.5+\frac{Q-1}{2}-i)$ is a constant depends on Q only.

Substitute (1b) into D_k^Q and normalize it with respect to $A_{-(Q-1)/2}^Q(0.5)$,

$$\frac{D_k^Q}{A^Q_{-\frac{Q-1}{2}}(0.5)} = (-1)^{k+\frac{Q-1}{2}} \bullet \frac{Q!}{(\frac{Q-1}{2}+k)!(\frac{Q+1}{2}-k)!} \qquad (8)$$

Let l = k + (Q-1)/2, or k = l – (Q-1)/2, hence (8) becomes

$$\frac{D_k^Q}{A^Q_{-\frac{Q-1}{2}}(0.5)} = (-1)^l \bullet \frac{Q!}{l!(Q-l)!} = (-1)^l \binom{Q}{l} \qquad (9)$$

Combine (6), (7) and (9), we have

$$\frac{x(0.5^-) - x(0.5^+)}{A^Q_{-\frac{Q-1}{2}}(0.5)} = \sum_{k=0}^{Q}(-1)^k \binom{Q}{k} x(k-\frac{Q-1}{2}) \equiv \Delta^Q x(n)\Big|_{n=\frac{Q+1}{2}} \approx 0 \qquad (10)$$

where Δ denotes the differentiation operation in discrete-time. The final approximation is reasonable if the original signal can be approximated with negligible error using a (Q-1)-degree polynomial. In other words, the difference between the left-hand limit and right-hand limit at t=0.5 is in the same order as polynomial interpolation error of any other time instant. Therefore, it can be concluded that the interpolation results are continuous if the interpolation approximation error can be neglected.

5. EXAMPLE

We assume that a received signal is BPSK modulated baseband signal being sampled at twice the symbol rate. The pulse shaping filter used is a raised-cosine filter with excess bandwidth of 40%. The received signal is limited to the bandwidth of [0, $0.35f_s$], where f_s is the sampling frequency. In theory, the original signal can be perfectly reconstructed by using an ideal reconstruction filter from two samples per symbol period.

The received signal is obtained at different sampling phases, with timing errors ranging from 0 to $0.5T_s$ from the optimum symbol and mid-symbol sampling points. These sampling phases correspond to different fractional intervals from 0 to 0.5. We then use Lagrange polynomial interpolation filters to recover the symbols at the symbol timing strobe. The approximation errors can be determined by calculating the mean square error (MSE) at these decision points for different values of sampling timing error [6]. No noise is added to the signal such that the interpolation is the only error source. The results of this simulation are shown in Fig. 7. When the timing errors are between $0.5T_s$ and T_s (when an even number of basepoints are used), or between $-0.5T_s$ and 0 (when an odd number of basepoints are used), the MSE results are identical to what have been shown for μ∈[0, 0.5]. In general, the higher order of a Lagrange interpolator, the lower MSE of the interpolation results are, because of their better image rejection. But for this test case, it is worth notice that an odd-order interpolator using more basepoints (Q=4) does not necessarily result in smaller MSE when compared to an even-order interpolator using fewer basepoints (Q=3).

6. CONCLUSION

Previous references concluded that a polynomial based interpolation filter must have even number of samples in its basepoint set in order to have linear phase filtering. A new method is proposed in this paper to construct linear phase Lagrange polynomial-based interpolation filter using odd number of basepoints. Although the impulse response of the conceptual analog reconstruction filter is not time-continuous when the hybrid analog/digital model is employed to analyze the interpolation process, it is proved that the interpolation results are time-continuous within the approximation error introduced by such interpolation.

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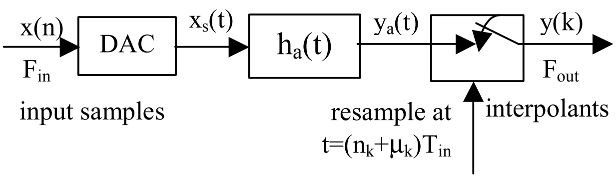

Fig. 1 The hybrid analog/digital model for interpolation

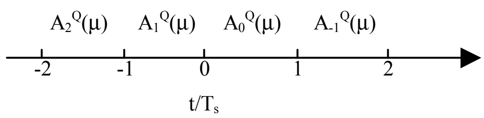

Fig. 2 Construct analog reconstruction filter based on Lagrange polynomial (Q=4)

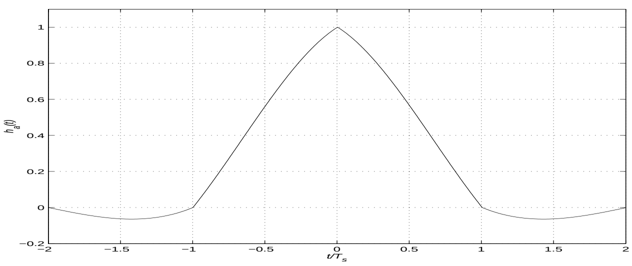

Fig. 3 Impulse response of cubic interpolator (Q=4)

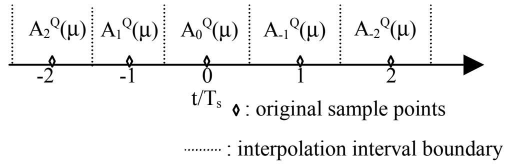

Fig. 4 Construct analog reconstruction filter based on proposed Lagrange polynomial method (Q=5)

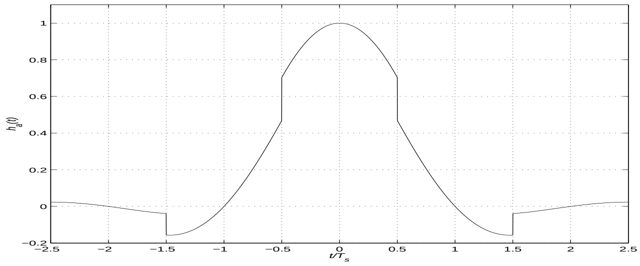

Fig. 5 Impulse response of 4th order interpolator (Q=5)

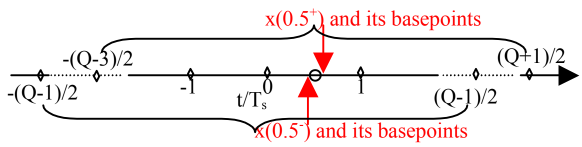

Fig. 6 Calculation of interpolants $x(0.5^-)$ and $x(0.5^+)$

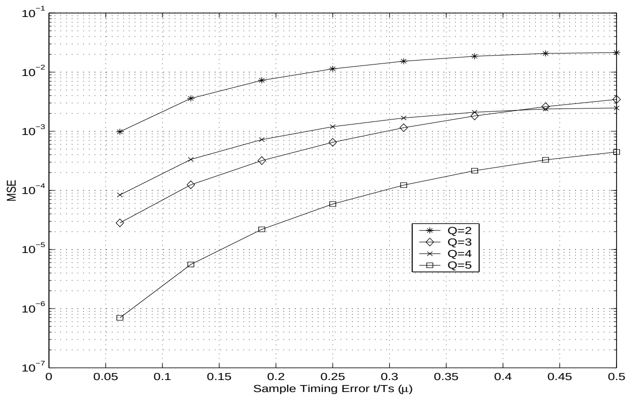

Fig. 7 MSE at symbol decision point using Lagrange interpolator of different orders