# LINEAR PHASE EQUIRIPPLE IIR DIGITAL FILTER DESIGN

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## ABSTRACT

In this paper, a design approach for stable IIR digital filters with equiripple magnitude in passband and stopband and linear phase in passband is presented. The design approach adopts a constant group delay all-pole filter in the denominator and a mirror image polynomial transfer function in the numerator. Starting with a suitable filter order and a constant group delay value for the denominator, the Remez exchange algorithm is used to aid the design of the equiripple passband and stopband numerator transfer function. Equiripple passband and stopband characteristic offers benefits of reduced width in transition band and lower filter order. Lowpass, highpass, bandpass, and bandstop filter design examples are given.

## 1. INTRODUCTION

Digital filters with linear phase or constant group delay property are desirable in applications where the time differences among the frequency components of a signal before and after filtering must be preserved. The non-existence of stability problem and simplicity of linear phase FIR digital filters are known. The drawbacks of a linear phase FIR digital filter include its higher group delay constant and higher filter-order as compared to its corresponding IIR digital filter, especially in applications requiring a filter with a narrow transition band. Design of IIR digital filters with linear phase or constant group delay characteristics is of practical interest.

An allpass approach for designing an IIR digital filter with a constant group delay characteristic is to design an additional allpass equalizer to equalize the group delay of the IIR digital filter over the passband of interest [1-2]. A linear programming approach [3] adopts linear programming technique to approximate prescribed magnitude and group delay of an IIR digital filter. This approach gives an optimal solution if it exists. In [2], the theory of eigen filters is extended to the design of an allpass filter that compensates for the nonlinear phase response of an IIR digital filter, as well as to the design of IIR digital filters for the simultaneous approximation of magnitude response and constant group delay. A nonlinear programming approach is to design an IIR digital filter that simultaneously approximates prescribed magnitude and group delay using nonlinear optimization techniques [2, 4]. In [4], a new nonlinear programming algorithm called the recursive quadratic programming is used to design such an IIR digital filter that yields shorter group delay response in examples than those of [2].

A mirror image polynomial approach involves the use of a mirror image type of transfer function [5-8]. In [6], the coefficients of an all-pole transfer function are first optimized in the least $p^{th}$ sense to approximate the constant group delay. Then the coefficients of a cascaded mirror image polynomial are optimized also in the least $p^{th}$ sense to approximate the prescribed magnitude response. An analytical procedure is presented in [7] to obtain an explicit form of an allpass transfer function that approximates a maximally flat group delay response. The coefficients of a cascaded numerator transfer function are then designed to approximate equiripple stopband magnitude response. Adopting the Thiran's constant group delay all-pole transfer function [5] as the denominator, new methods [8] for the zero-determination of a cascaded numerator transfer function are presented to achieve maximally flat passband and equiripple stopband magnitude response. Similar techniques for designing 2-dimensional IIR digital filters can be found in [9-11].

Recently, a method [12-13] is advanced for designing IIR digital filters approximating maximally flat magnitude and group delay characteristics in the passband. The method can achieve an equiripple approximation to a constant group delay in the passband and an equiripple magnitude response in the stopband. The method can also achieve a magnitude response in the passband and stopband with different degrees of flatness at w=0 and $\pi$. In the design method, the Thiran's all-pole transfer function [5] is used to approximate a constant group delay in the maximally flat or equiripple sense. A cascaded mirror image polynomial is used to approximate the magnitude response requirements. The coefficients of this numerator transfer function can be obtained by an analytical method, when a maximally flat magnitude approximation at w=0 or a magnitude response with different degrees of flatness at w=0 and w=$\pi$ is required.

In [14], a class of approximately linear phase recursive digital filters composed of two allpass sections is designed using optimization. By choosing one of the allpass sections to be a pure delay network [15], the remaining allpass section is optimized to approximate prescribed magnitude and linear phase requirements. These are some of the typical methods for designing IIR digital filters meeting prescribed magnitude and group delay requirements. A review of the design techniques of both 1-dimensional and 2-dimensional IIR digital filters satisfying prescribed magnitude and constant group delay specifications can be found in [16].

In this paper, we present the results of a design approach for approximating equiripple passband and stopband magnitude response with a constant group delay in the passband of an IIR digital filter. The method adopts the combination of a mirror image numerator polynomial to approximate equiripple magnitude response in the passband and stopband and an all-pole transfer function to provide a constant group delay passband to form an IIR digital filter. The Remez exchange algorithm [17] is used to facilitate the equiripple filter design procedure. As the Thiran's all-pole transfer function [5] adopted in this approach is a lowpass filter, simple linear-phase-preserved transformations of z are used to aid highpass, bandpass, and bandstop digital filter design.

## 2. FILTER FORMAT

The transfer function of an IIR digital filter can be represented as:

$$H(z^{-1})=N(z^{-1})/D(z^{-1}) \qquad (1)$$

In the design method, the Thiran's all-pole group delay filter is adopted as the denominator transfer function, $D(z^{-1})$, whereas a mirror image polynomial is adopted as the numerator transfer function, $N(z^{-1})$. The combination of a constant group delay passband filter with a real-valued even-order mirror image polynomial yields a digital filter with a constant group delay in the passband.

In [5], an all-pole transfer function that exhibits a constant group delay characteristic is presented. The transfer function is given by:

$$\frac{1}{D(z^{-1})} = \frac{\left\{\frac{2T!}{T!}\frac{1}{\prod_{i=T+1}^{2T}(2\tau+i)}\right\}}{\sum_{k=0}^{T}\left[(-1)^k\binom{T}{k}\prod_{i=0}^{T}\frac{2\tau+i}{2\tau+k+i}\right]z^{-k}} \qquad (2)$$

T and $\tau$ represent, respectively, the filter-order and the group delay value of the transfer function. This all-pole digital filter can be shown to be stable for all finite positive values of $\tau$.

In order to maintain the constant group delay characteristic of the all-pole digital filter, a mirror image polynomial with an even filter order is used which can be expressed as:

$$N_m(z^{-1}) = b_0 + b_1\left(\frac{z+z^{-1}}{2}\right) + \ldots + b_q\left(\frac{z^{q}+z^{-q}}{2}\right) \qquad (3)$$

(3) is a non-causal transfer function and can be realized as a causal transfer function by expressing the numerator transfer function as:

$$N(z^{-1}) = z^{-q} N_m(z^{-1}) \qquad (4)$$

By combining (2) and (4), we obtain a digital filter as represented by (1) with a constant group delay of $q+\tau$.

Substituting $z = e^{jwT}$ into (3), the frequency response of the mirror image polynomial is given by:

$$N_m(e^{-jwT}) = b_0 + b_1\cos(wT) + b_2\cos(2wT) + \ldots + b_q\cos(qwT) \qquad (5)$$

## 3. EQUIRIPPLE FILTER DESIGN

From (1), the real transfer function of the digital filter can be defined as:

$$H_r(e^{-jwT}) = N_m(e^{-jwT}) / |D(e^{-jwT})| \qquad (6)$$

Hence, from (4)-(6),

$$b_0 + b_1\cos(wiT) + b_2\cos(2wiT) + \ldots + b_q\cos(qwiT)$$
$$= Hr(e^{-jwiT})\,|D(e^{-jwiT})| \qquad (7)$$

At any frequency wiT, given a desired magnitude response $H_d(e^{-jwiT})$ and a specified $|D(e^{-jwiT})|$, (7) can be used as one of the equations to solve the coefficients of the mirror image polynomial.

For equiripple passband and stopband design, assuming $\delta p$ and $\delta s$ are the peak values of the ripples in the passband and stopband of a digital filter. The passband peak ripples are defined as:

$$|H_d(e^{-jwpiT})| - N_m(e^{-jwpiT}) / |D_d(e^{-jwpiT})| = \pm(-1)^i\,\delta p \qquad (8a)$$

$$\text{for } w_{pu} < w_{p1} < w_{p2} < w_{p3} < \ldots < w_{pP} < w_{pv} \qquad (8b)$$

Similarly, the stopband peak ripples are defined as:

$$|H_d(e^{-jwsiT})| - N_m(e^{-jwsiT}) / |D_d(e^{-jwsiT})| = \pm(-1)^i\,\delta s \qquad (9a)$$

$$\text{for } w_{su} < w_{s1} < w_{s2} < w_{s3} < \ldots < w_{sS} < w_{sv} \qquad (9b)$$

(7)-(9) can be solved with the aid of the Remez exchange algorithm to design a digital filter with equiripple passband and stopband characteristics.

## 4. FILTER EXAMPLES

Four equiripple passband and stopband IIR digital filters each with a constant group delay of 1 in the denominator are designed. The specifications and results of the 4 filter examples are summarized in Tables 1-2. The magnitude responses (with enlarged magnitude responses in the passbands) and the phase responses of the four filter examples are shown respectively in Figs. 1-2, 3-4, 5-6, and 7-8.

Based on the specifications (L=5) of the lowpass filter given in the Fig. 5 of [13], our design yields a numerator transfer function with a filter order 24 instead of 26 as specified in [13]. The passband and stopband peak ripples, and the total group delay of our designed filter are, respectively, -37.52dB, -40.08 dB, and 13. This result is due to the adoption of an equiripple passband instead of a maximally flat passband design strategy adopted in [13]. In the highpass filter example, a lowpass filter with passband and stopband cutoff frequencies '$\pi$-wp' and '$\pi$-ws' is first designed. By taking the z to –z transformation to the lowpass filter, the highpass IIR digital filter is obtained. In the bandpass filter example, the denominator transfer function is first subject to z to $z^4$ transformation to obtain appropriate constant group delay and magnitude responses. The Remez algorithm is then applied to design the numerator transfer function of the bandpass filter. Alternatively, an appropriate Thiran's all-pole lowpass filter could be used directly as the denominator of the bandpass filter without applying the z to $z^4$ transformation. In the bandstop filter example, a prototype lowpass filter of passband and stopband cutoff frequencies 2wp1T and 2ws1T is first designed and the filter is obtained by z to $z^2$ transformation.

| Filters | LP | HP | BP | BS |
|---|---|---|---|---|
| Wp1T,Wp2T | 0.12 | 0.79 | 0.2, 0.3 | 0.14, 0.86 |
| Ws1T,Ws2T | 0.17 | 0.74 | 0.17, 0.33 | 0.165, 0.835 |
| Nd | 5 | 7 | 9 | 9 |
| Nn (Nc) | 79 (40) | 81 (41) | 97 (49) | 77 (39) |
| PB1, PB2 | 5 | 9 | 5 | 11, 11 |
| SB1, SB2 | 33 | 30 | 7, 32 | 52 |
| Gd | 40 | 41 | 52 | 78 |
| Iter | 5 | 6 | 8 | 5 |

**Table 1.** Summary of specifications and results of 4 filter examples. Keys: Wp1T, Wp2T – Normalized passband cutoff frequencies/π; Ws1T, Ws2T – Normalized stopband cutoff frequencies/π; Nd – Number of denominator coefficients; Nn – Number of numerator coefficients; Nc - Number of distinct numerator coefficients; PB1, PB2 - Number of passband peak ripples; SB1, SB2 - Number of stopband peak ripples; Gd - Total passband group delay; Iter – Number of iterations to converge.

| Filters | LP | HP | BP | BS |
|---|---|---|---|---|
| MPPR | -37.37 | -40.51 | -32.84 | -35.72, -35.72 |
| MSPR | -44.74 | -43.23 | -33.19, -32.85 | -44.81 |

**Table 2.** Summary of maximum passband peak ripple (MPPR) and maximum stopband peak ripple (MSPR) (in dB) of 4 filter examples.

## 5. SUMMARY

In this paper, we have described a design approach for stable IIR digital filters with equiripple magnitude response in passband and stopband and linear phase response in passband. With equiripple passband and stopband magnitude responses, the filter-order and the transition width of an IIR digital filter can be reduced. In practice, the level of the passband ripple peaks can be designed to meet the specific requirement of an individual application.

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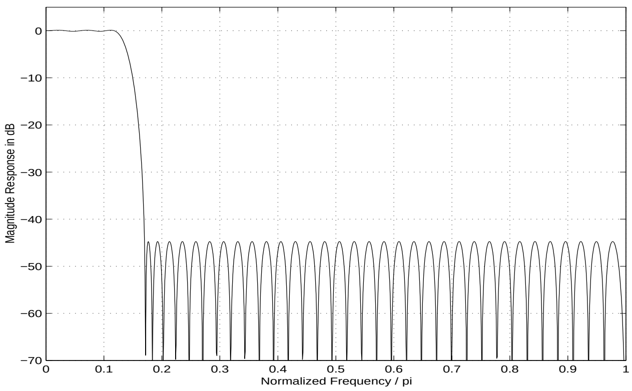


**Fig. 1.** Magnitude response of Filter 1.

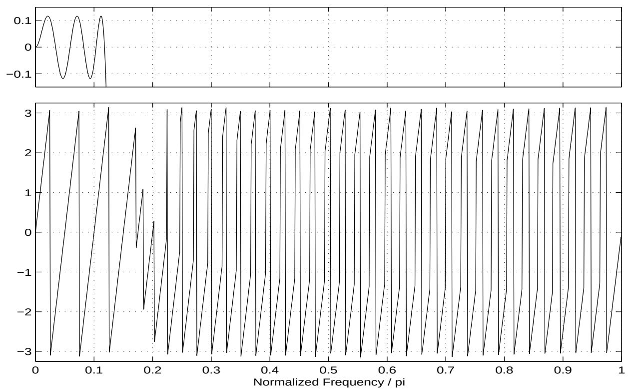


**Fig. 2.** Magnitude (dB) and phase (radians) responses of Filter 1.

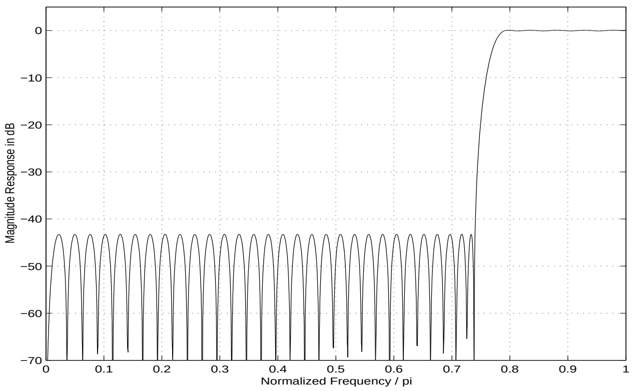


**Fig. 3.** Magnitude response of Filter 2.

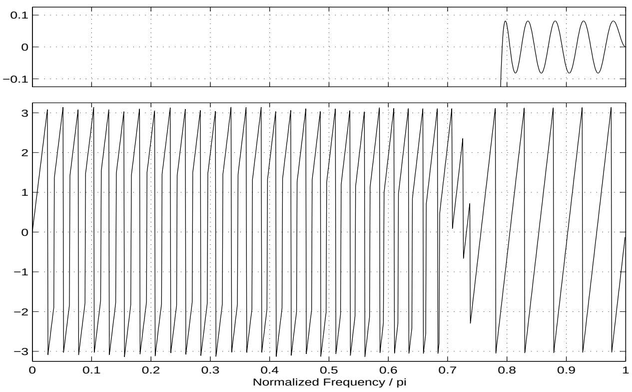


**Fig. 4.** Magnitude (dB) and phase (radians) responses of Filter 2.

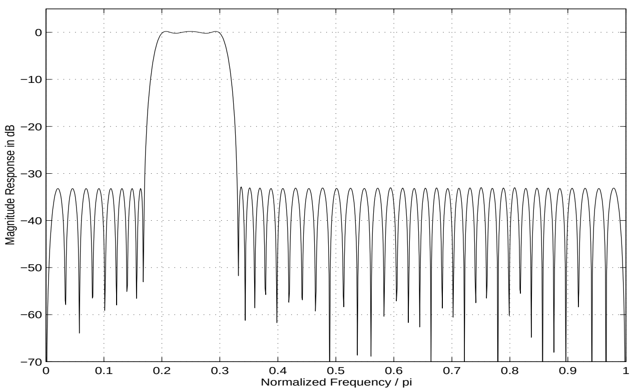


**Fig. 5.** Magnitude response of Filter 3.

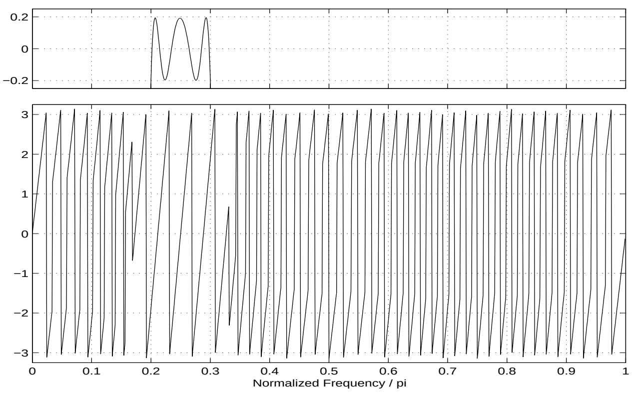


**Fig. 6.** Magnitude (dB) and phase (radians) responses of Filter 3.

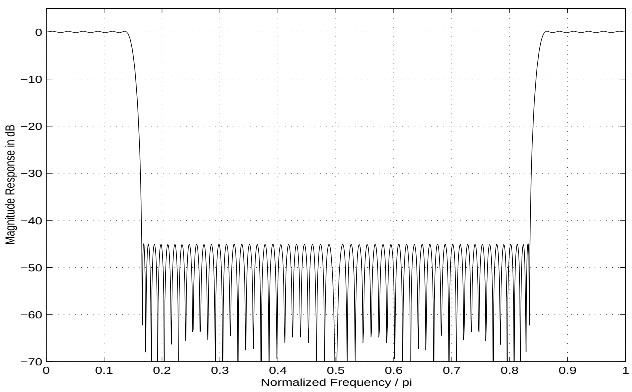


**Fig. 7.** Magnitude response of Filter 4.

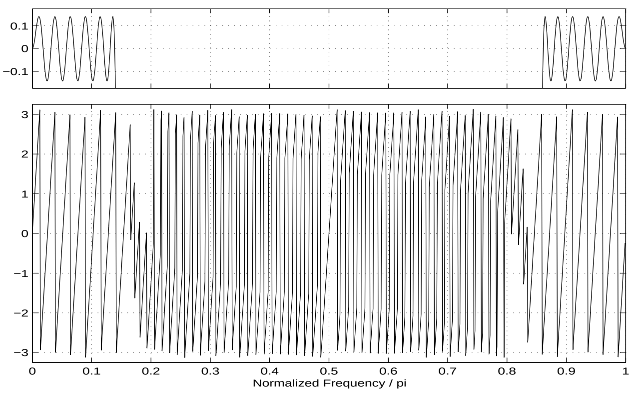


**Fig. 8.** Magnitude (dB) and phase (radians) responses of Filter 4.