DESIGN OF ROBUST IIR MAGNITUDE FILTERS VIA SEMIDEFINITE PROGRAMMING

Yonghong Liu and Zhi-Quan Luo

Department of Electrical and Computer Engineering
McMaster University, Hamilton, Ontario L8S 4K1, Canada

ABSTRACT

In this paper we consider the design of lowpass infinite impulse response (IIR) magnitude filters which are robust against the implementation error. It is shown that the design problem can be cast as a quasiconvex problem with a set of linear matrix inequality (LMI) constraints and the autocorrelation sequences of the filter coefficients as the design variables. The relation between the norm error of autocorrelation sequences and that of filter coefficients is derived, and the issue of filter stability is addressed by deriving a lower bound on the distance from the pole to the unit circle. Simulation results show that our designed filter is immune from the errors caused by finite precision implementation. The method can also be used in similar highpass and bandpass IIR filter design.

1. INTRODUCTION

In hardware implementation of digital filters, each filter tap (coefficient) can only be represented as a finite word length number. As a result, an optimal filter designed using real number formulation must be rounded (or truncated) before digital implementation. However, such rounding procedure may result in degradation of filter performance or even render the filter unstable. Alternatively, we may use mixed-integer linear programming formulation to design optimum filters with sum of power of two coefficients [1]–[5]. However, this problem formulation is not convex and suffers from high computational complexity. In a recent paper [6], Lu proposed a method by relaxing the design to a semidefinite programming (SDP) problem, which can be solved using efficient interior-point algorithms with polynomial-time complexity. However, this approach does not explicitly address the issue of filter robustness against rounding errors.

In this paper, we present a new approach to the design of IIR magnitude filters which are robust against the implementation (or rounding) errors. In section 3, we show that such robust design problem can be formulated as a quasiconvex problem which can be solved via SDP feasibility problems without any approximation. A design example is given in section 4 to illustrate the performance of the proposed technique.

The notation we use in this paper is as follows:

- $x \succeq 0$: vector x is an autocorrelation sequence;
- $X \succeq 0$: matrix X is positive semidefinite;
- **I**: identity matrix of appropriate size;
- **0**: zero matrix of appropriate size.

This research is supported in part by the Natural Sciences and Engineering Research Council of Canada, Grant No. OPG0090391 and by the Canada Research Chair Program.

2. PROBLEM STATEMENT

The transfer function of an IIR filter is given by

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}} \triangleq \frac{A(z)}{B(z)}$$

where we assume the filter coefficients to be real valued. We shall use $a = [a_0, \cdots, a_n]^T \in R^{n+1}$ and $b = [b_0, \cdots, b_m]^T \in R^{m+1}$ to denote the filter tap vectors.

In this paper, we consider the typical problem of designing a lowpass IIR filter with maximum stopband attenuation, subject to spectral mask constraint on the magnitude response in the passband. This design problem can be expressed as [7]

$$\begin{array}{ll} \text{minimize} & \sup_{\omega \in [\omega_s, \pi]} |H(e^{j\omega})| \\ \text{subject to} & 1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p, \ \omega \in [0, \omega_p], \\ & b^T b = 1, \end{array} \quad (1)$$

where δ_p is the passband deviation, ω_s the stopband edge, and ω_p the passband edge. The equality constraint is used to normalize the solution.

Notice that the optimal design given by (1) will satisfy the spectral mask constraint. However, after quantization (which is necessary for digital implementation), the resulting filter design may violate the spectral mask constraint, or even become unstable. This motivates us to consider introducing robustness in the above design formulation. We address this issue in the next section.

3. PROBLEM FORMULATION

3.1. Preliminary Formulation

Problem (1) is not convex in the variables a and b. However, it was pointed out in [7]–[11] that this problem can be reformulated as a quasiconvex problem if the autocorrelation coefficients of a and b are used as variables.

Define $u = [u_0, \cdots, u_n]^T \in R^{n+1}$ and $v = [v_0, \cdots, v_m]^T \in R^{m+1}$ as the autocorrelation coefficients of a and b, and let

$$U(\omega) \triangleq u_0 + 2u_1 \cos\omega + \cdots + 2u_n \cos n\omega,$$

$$V(\omega) \triangleq v_0 + 2v_1 \cos\omega + \cdots + 2v_m \cos m\omega,$$

then it is easily shown that problem (1) can be expressed as

$$\begin{array}{lll} \text{minimize} & t & \\ \text{subject to} & U(\omega) \leq (1+\delta_p)^2 V(\omega), & \omega \in [0, \omega_p], \\ & (1-\delta_p)^2 V(\omega) \leq U(\omega), & \omega \in [0, \omega_p], \\ & 0 \leq U(\omega) \leq t V(\omega), & \omega \in [\omega_s, \pi], \\ & V(\omega) \geq 0, \quad U(\omega) \geq 0, & \omega \in [0, \pi], \\ & v_0 = 1, & \end{array} \quad (2)$$

where the non-negativity constraints $U(\omega) \geq 0$ and $V(\omega) \geq 0$ are imposed to guarantee the existence of a and b [8]. This is a quasiconvex problem with variables t, u, and v, and can be solved via binary search on t, $t \in R$. Once a solution of (2) is found, an IIR filter can be obtained via spectral factorization.

3.2. Stability Considerations

For an IIR filter to be stable, all the poles must lie inside the unit circle. Once an optimal solution u and v is obtained, choosing minimum phase spectral factors yields an IIR filter with no poles and zeros outside the unit circle. However, in practice, a system with a pole on the unit circle is also regarded as unstable or potentially unstable unless it coincides with a zero on the unit circle, since a minor disturbance or error will push the system into instability. Consequently, we set a stability margin $\varepsilon_1 > 0$ such that $V(\omega) \geq \varepsilon_1$ for all ω, which implies that none of the poles can be on the unit circle when minimum phase spectral factors are extracted.

Let $R(z) = z^m B(z)$, and suppose $\lambda \in C$ is a root of $R(z)$ with the maximum magnitude, then from the theorem of Lagrange mean and the fact that $\left|R(e^{j\omega})\right|^2 = V(\omega) \geq \varepsilon_1$, it immediately follows that the minimum distance d_{min} from the poles to the unit circle can be lower bounded as

$$d_{\min} = |1 - |\lambda|| \geq \frac{\sqrt{\varepsilon_1}}{m|b_0| + (m-1)|b_1| + \cdots + |b_{m-1}|}. \quad (3)$$

3.3. Intermediate Formulation

For a vector $x = [x_0, x_1, \ldots, x_n]^T \in R^{n+1}$, the following properties are proven in [7, 10]:

1. x satisfies the constraint of the form

$$X(\omega) = x_0 + 2\sum_{k=1}^{n} x_k \cos k\omega \geq 0, \quad \omega \in [\alpha, \beta] \quad (4)$$

   if and only if

$$L(\alpha, \beta)x \succeq 0,$$

   where matrix $L(\alpha,\beta) \in R^{(n+1)\times(n+1)}$ depends only on α and β and can be constructed based on the recursion of Chebyshev Polynomials;

2. $x \succeq 0$ if and only if there exists a $P = P^T \in R^{n\times n}$ such that

$$\begin{bmatrix} P - M^T PM & \tilde{x} - M^T PN \\ \tilde{x}^T - N^T PM & x_0 - N^T PN \end{bmatrix} \succeq 0 \quad (5)$$

   with

$$M = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ 0 & \mathbf{0} \end{bmatrix} \in R^{n\times n}, N = [0, \cdots, 0, 1]^T \in R^n, \quad (6)$$

   and $\tilde{x} = [x_n, x_{n-1}, \cdots, x_1]^T$.

Without loss of generality, we suppose $n \geq m$. Notice that the constraints in (2) have the form of (4). Defining matrices

$$L_1 = L(0, \omega_p) \in R^{(n+1)\times(n+1)},$$

$$L_2 = L(\omega_s, \pi) \in R^{(n+1)\times(n+1)},$$

$$E_x \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \end{bmatrix} \in R^{n\times(n+1)}, \quad (7)$$

and

$$e \triangleq [1, 0, \cdots, 0]^T \in R^{n+1}, \quad (8)$$

applying the above two properties, through some simple derivations, we rewrite problem (2) and stability considerations as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} S_i & C_i \\ C_i^T & W_i \end{bmatrix} \succeq 0, \quad i = 1, \cdots, 5, \\ & P_i = P_i^T \in R^{n\times n}, \quad i = 1, 2, 3, 5, \\ & P_4 = P_4^T \in R^{m\times m}, \\ & v_0 = 1 \end{array} \quad (9)$$

with

$$\begin{aligned} S_i &= P_i - M_i^T P_i M_i, \quad i = 1, \cdots, 5, \\ C_1 &= (1+\delta_p)^2 E_{x1} L_{1v} v - E_{x1} L_{1u} u - M_1^T P_1 N_1, \\ C_2 &= E_{x2} L_{1u} u - (1-\delta_p)^2 E_{x2} L_{1v} v - M_2^T P_2 N_2, \\ C_3 &= t E_{x3} L_{2v} v - E_{x3} L_{2u} u - M_3^T P_3 N_3, \\ C_4 &= E_{x4}(v - \varepsilon_1 e_2) - M_4^T P_4 N_4, \\ C_5 &= E_{x5}(u - \varepsilon_2 e_3) - M_5^T P_5 N_5, \\ W_1 &= (1+\delta_p)^2 e_1^T L_{1v} v - e_1^T L_{1u} u - N_1^T P_1 N_1, \\ W_2 &= e_1^T L_{1u} u - (1-\delta_p)^2 e_1^T L_{1v} v - N_2^T P_2 N_2, \\ W_3 &= t e_1^T L_{2v} v - e_1^T L_{2u} u - N_3^T P_3 N_3, \\ W_4 &= e_2^T (v - \varepsilon_1 e_2) - N_4^T P_4 N_4, \\ W_5 &= e_3^T (u - \varepsilon_2 e_3) - N_5^T P_5 N_5, \end{aligned}$$

where L_{1v} is the matrix consisting of the first $m+1$ columns of matrix L_1, L_{2v} the first $m+1$ columns of matrix L_2, $L_{1u} = L_1$, $L_{2u} = L_2$, M_i, N_i, E_{xi}, and e_i are matrices and vectors with sizes inferred from context and structures defined by (6), (7), and (8), respectively, and we let $U(\omega) \geq \varepsilon_2 > 0$ to simplify the spectral factorization. Obviously, some of these data matrices, depending on their dimensions, are identical. Notice that problem (9) is a quasiconvex problem with a set of LMI constraints. In particular, for each fixed t, the constraints of (9) become linear matrix inequalities, so the feasibility of the constraint set can be checked efficiently using interior-point methods for each fixed t. With this procedure to check feasibility, the minimum t for (9) can be found via the standard binary search technique.

3.4. Robustness Considerations

In the presence of rounding errors, we must make sure the rounded solution remains stable and still respects the spectral mask constraint. We will model the rounding error explicitly to avoid the potential instability and to ensure the satisfaction of the spectral mask constraints after rounding.

If we let matrix $F_1(P_1, u, v) \triangleq \begin{bmatrix} S_1 & C_1 \\ C_1^T & W_1 \end{bmatrix}$, then under the robustness considerations, the constraint $F_1(P_1, u, v) \succeq 0$ is changed to

$$F_1(P_1, u + \triangle u, v + \triangle v) \succeq 0 \quad (10)$$

for all $\left\| \begin{smallmatrix} \triangle u \\ \triangle v \end{smallmatrix} \right\| \leq \delta_1$, where δ_1 is the norm error of variables u and v, and $\delta_1 > 0$.

Given $\left\| \begin{smallmatrix} \triangle a \\ \triangle b \end{smallmatrix} \right\| \leq \delta$, by Cauchy-Schwarz inequality and the definition of u and v, it is easily shown that

$$\triangle u_i \leq 2\|a\|\delta + \delta^2, i = 0, 1, \cdots, n, \tag{11}$$

and

$$\triangle v_i \leq 2\delta + \delta^2, i = 0, 1, \cdots, m. \tag{12}$$

By Parseval's Relation, we get

$$\|a\| \leq 1 + \delta_p. \tag{13}$$

Combining (11), (12) and (13) yields

$$\left\| \begin{matrix} \triangle u \\ \triangle v \end{matrix} \right\| \leq \delta_1 \tag{14}$$

with $\delta_1 = \sqrt{(n+1)[2(1+\delta_p)\delta + \delta^2]^2 + (m+1)(2\delta+\delta^2)^2}$.

Similarly, the other inequality constraints in (9) can also be changed to constraints of form (10).

3.5. Final Formulation

Luo *et al.* in [12] has recently proven that the data matrices (S, C, D, W, G, H, Q) satisfy the following condition

$$\begin{bmatrix} S & C + DX \\ (C+DX)^T & W + X^T G + G^T X + X^T H X \end{bmatrix} \succeq 0$$

for any X satisfying $\mathbf{I} - X^T Q X \succeq 0$ if and only if there exists a $\gamma \geq 0$ such that

$$\begin{bmatrix} S & C & D \\ C^T & W & G^T \\ D^T & G & H \end{bmatrix} - \gamma \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -Q \end{bmatrix} \succeq 0.$$

By applying the above theorem, the constraint of form (10) can be transformed into an LMI constraint with some auxiliary variable, and therefore the problem (9) and robustness considerations can be rewritten as

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & \begin{bmatrix} S_i & C_i & D_i \\ C_i^T & W_i & G_i^T \\ D_i^T & G_i & \mathbf{0} \end{bmatrix} - \gamma_i \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \delta_1^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \succeq 0 \\ & \gamma_i \geq 0, \qquad i = 1, \cdots, 5, \\ & P_i = P_i^T \in R^{n \times n}, \quad i = 1, 2, 3, 5, \\ & P_4 = P_4^T \in R^{m \times m}, \\ & v_0 = 1 \end{aligned} \tag{15}$$

with

$$D_1 = [\ -E_{x1}L_{1u} \quad (1+\delta_p)^2 E_{x1} L_{1v}\ ],$$

$$D_2 = [\ E_{x1}L_{1u} \quad -(1-\delta_p)^2 E_{x1} L_{1v}\ ],$$

$$D_3 = [\ -E_{x3}L_{2u} \quad tE_{x3}L_{2v}\ ],\ D_4 = [\ \mathbf{0} \quad E_{x4}],$$

$$D_5 = [\ E_{x3} \quad \mathbf{0}\ ],\ G_1^T = 0.5[\ -e_1^T L_{1u} \quad (1+\delta_p)^2 e_1^T L_{1v}\ ],$$

$$G_2^T = 0.5[\ e_1^T L_{1u} \quad -(1-\delta_p)^2 e_1^T L_{1v}\ ], G_4^T = 0.5[\ \mathbf{0} \quad e_2^T\ ],$$

$$G_3^T = 0.5[\ -e_1^T L_{2u} \quad te_1^T L_{2v}\ ],\ G_5^T = 0.5[\ e_3^T \quad \mathbf{0}\ ].$$

Problem (15) can be solved via bisection on t, *i.e.* solving a sequence of SDP feasibility problems. Then, we use minimum phase spectral factorization to recover filter coefficients a and b from the autocorrelation coefficients u and v.

Obviously, the techniques described above are readily extended to related problems, such as robust highpass and bandpass IIR magnitude filter design.

4. SIMULATION RESULTS

Two lowpass IIR digital filters were designed to illustrate the performance of our proposed technique: one designed by the proposed method with norm error of filter coefficients set to $\delta = 10^{-6}$, the other without taking robustness issue into account. Both satisfy the same set of specifications: passband edge $\omega_p = 0.15\pi$, stopband edge $\omega_s = 0.30\pi$, maximum passband deviation $\delta_p = 0.01$, numerator degree $n = 5$, denominator degree $m = 4$, stability margin $\varepsilon_1 = 10^{-8}$, and $\varepsilon_2 = 10^{-8}$. The SDP optimization was implemented in SeDuMi Interface 1.01 [13] and SeDuMI 1.05 [14] with MATLAB 6.1.

The simulation results are shown in Figs. 1 and 2, respectively. The solid curves correspond to the optimal filters. The maximum stopband attenuation is 0.029444 or 30.62dB for the optimal robust filter vs. 0.0032434 or 49.78dB for the optimal nonrobust filter, which implies that the robust optimization method trades off the stopband performance for robustness.

As shown in Figs. 1 and 2, when the filter coefficients are subject to perturbation $\delta = 10^{-6}$, little change can be found in the magnitude response of the robust filter and the spectral mask constraint is still satisfied. In contrast, the same perturbation results in violation of the passband spectral mask specification by the nonrobust filter. As the size of rounding perturbation increases, our robust filter remains immune from the perturbation $\delta \leq 6.0 \times 10^{-5}$, which is larger than the quantization error produced by 16 bit truncation of each filter coefficient. However, the violation of spectral mask constaint by the nonrobust filter becomes more pronounced with increasing δ. Meanwhile, pole-zero diagrams in Figs. 1 and 2 indicate that both filters remain stable when the filter coefficients are subject to different perturbations from 10^{-6} to 6.0×10^{-5}.

Our robust filter exhibits extra robustness in the simulations, as shown in Fig. 1. This is likely due to the bounds in (11), (12), and (14) are not tight.

5. CONCLUSIONS

We have proposed a new technique for the design of IIR digital magnitude filters which are immune from the implementation error. The design problem is formulated as a quasiconvex SDP problem and hence can be solved efficiently and exactly via interior-point methods. We demonstrated the effectiveness of our robust design technique by comparing with the nonrobust method. The technique can be extended to similar highpass and bandpass magnitude filter design. Our design technique can also be used in FIR case. All that is needed is to ignore the design variable b and the stability constraint.

6. REFERENCES

[1] Y.C. Lim, S.R. Parker, and A.G. Constantinides, "Finite wordlength FIR filter design using integer programming over

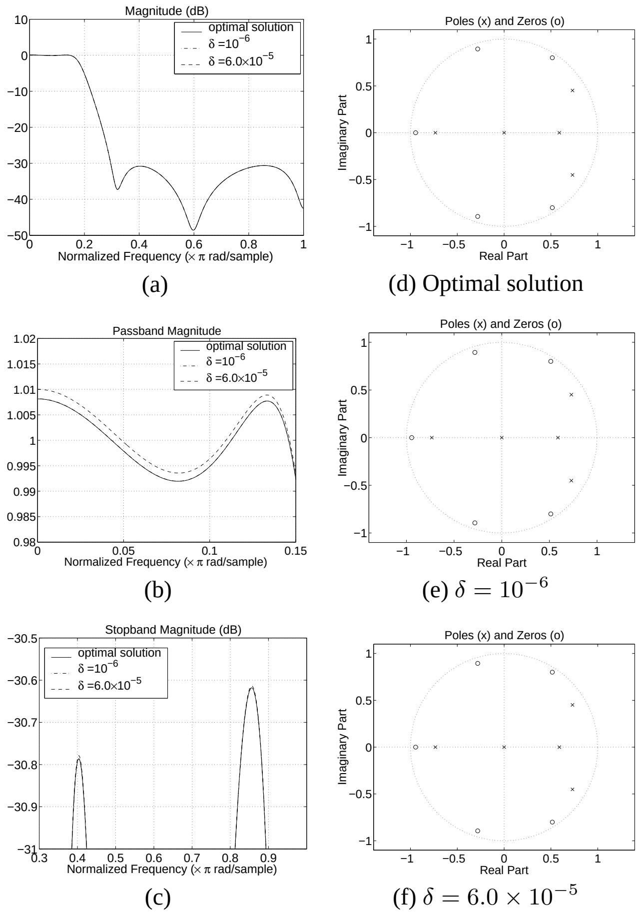

Fig. 1. Performances of robust IIR filter subject to different coefficient perturbations.

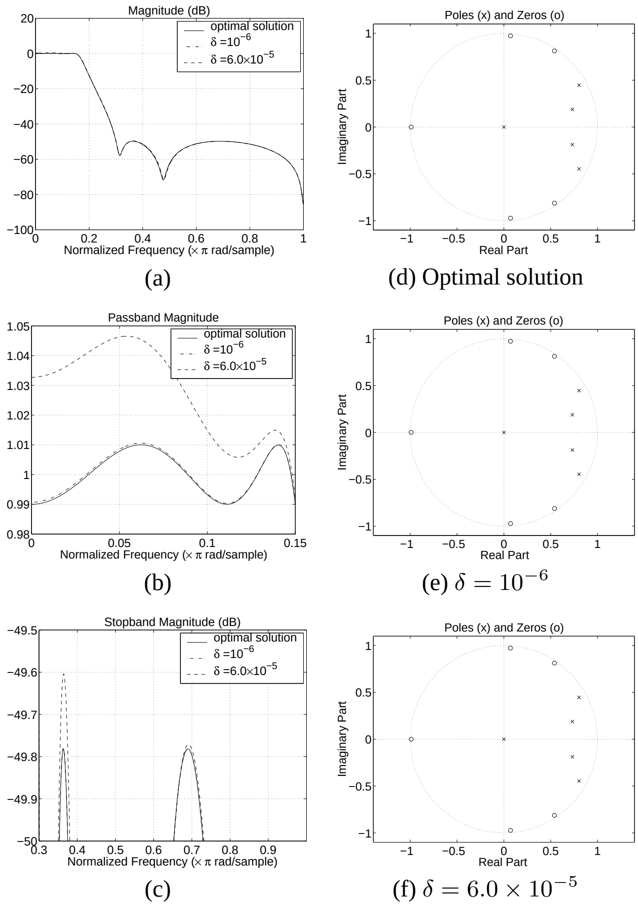

Fig. 2. Performances of nonrobust IIR filter subject to different coefficient perturbations.

a discrete coefficient space," *IEEE Trans. ASSP*, vol. 30, pp. 661–664, Aug. 1982.

[2] Y.C. Lim and S.R. Parker, "FIR filter design over a discrete power-of-two coefficient space," *IEEE Trans. ASSP*, vol. 31, pp. 583–591, June 1983.

[3] P. Siohan and A. Benslimane, "Finite precision design of optimal linear phase 2-D FIR digital filters," *IEEE Trans. Circuits Syst.*, vol. 36, pp. 11–22, Jan. 1989.

[4] Y.C. Lim, "Design of discrete-coefficient-value linear phase FIR filters with optimum normalized peak ripple magnitude," *IEEE Trans. Circuits Syst.*, vol. 37, pp. 1480–1486, Dec. 1990.

[5] Y.C. Lim, R. Yang, D. Li, and J. Song, "Signed power-of-two (SPT) term allocation scheme for the design of digital filters," *IEEE Trans. Circuits Syst. II: Analog and Digital Signal Processing*, vol. 46, pp. 577–584, May 1999.

[6] W.-S. Lu, "Design of FIR filters with discrete coefficients: a semidefinite programming relaxation approach," *2001 IEEE ISCAS 2001*, vol. 2, pp. 297–300, 2001.

[7] B. Alkire and L. Vandenberghe, "Convex optimization problems involving finite autocorrelation sequences," submitted to Mathematical Programming, Series B, January 5, 2001.

[8] Shao-Po Wu, Stephen Boyd, and Lieven Vandenberghe, "FIR filter design via spectral factorization and convex optimization," in *Applied and Computational Control, Signals and Circuits*, B. Datta, Ed., pp. 219–250, Birkhauser, 1998.

[9] T.N. Davidson, Z.Q. Luo, and K.M. Wong, "Orthogonal pulse shape design via semidefinite programming," in *Proc. 1999 IEEE ICASSP*, vol. 5, pp. 2651–2654, 1999.

[10] T.N.Davidson, Z.Q. Luo, and J.F.Sturm, "Linear matrix inequality formulation of spectral mask constraints," in *Proc. 2001 IEEE ICASSP*, vol. 6, pp. 3813–3816, 2001.

[11] Brien Alkire and Lieven Vandenberghe, "Interior-point methods for magnitude filter design," in *Proc. 2001 IEEE ICASSP*, vol. 6, pp. 3821–3824, 2001.

[12] Zhi-Quan Luo, Jos F. Sturm, Shuzhong Zhang, "Cones of nonnegative mappings — complexity and applications," Technical Report, Dept. of Electrical and Computer Engineering, McMaster University, 2002.

[13] Dimitri Peaucelle, Didier Henrion, and Yann Labit, "User's guide for SeDuMi Interface 1.01".

[14] Jos F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones (updated for version 1.05)".