A Design Method of Minimum Phase FIR Filters with Complex Coefficients

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Abstract: In this paper, we present a directly design method of minimum phase FIR filters with complex coefficients using successive projections (SP) method. In general, it is necessary to restrict the zeros of transfer function inside the unit circle or on the unit circle to directly obtain the transfer function of minimum phase FIR filters. We use the Rouche's theorem for coefficients update algorithm in SP method to restrict the zeros of the transfer function inside the unit circle and on the unit circle. Therefore, the transfer function of minimum phase FIR filter is obtained without factorization by using the proposed method. Moreover, the proposed method is also possible to obtain the transfer function of real coefficients by considering imaginary part of complex coefficients as the zero. The usefulness of the proposed method is verified through some examples.

1. Introduction

FIR digital filters are important in the field of waveform transmission and image processing in which phase distortion becomes a problem, because the filter with the perfect linear phase characteristics can be easily realized. However, in speech processing systems, FIR filters with linear phase characteristics are undesirable because of unacceptable high delays. Hence, the minimum phase FIR filters are highly attractive in this case. In addition, the minimum phase FIR filters with complex coefficients are also necessary to implement AM-SSB (Amplitude Modulation Single Side Band) which is a kind of amplitude modulation used in the telephone system.

Several methods for designing minimum phase FIR filters with real coefficients have been proposed previously [1]-[4]. In [1], *Hurrmann* and *Schuessler* (HS method) proposed a design method based on the Remez algorithm [5]. They first design a linear phase FIR filter with an equiripple characteristics as a prototype filter using Remez algorithm. Then, the obtained transfer function is factored by polynomial factorization to obtain the minimum phase FIR filters. Although the designed filters are theoretically optimal, a serious ill-condition problem often occurs during the polynomial factorization. In [2], *Ueda* et.al. solved these problems in numerical operations by increasing the gain level by little more amount in the comparison with HS method (modified HS method). However, these methods must design the filter with the twice order of the desired minimum phase FIR filter, since the desired minimum phase FIR filter is obtained by the factorization of the transfer function of the linear phase FIR filter. Therefore, these methods are a limit to design the transfer function of filter with 100th order. In addition, these methods can not handle filters with more than tow different gain level in each stopband, for example, bandpass filter with different gain level in their stopbands. In [3], *Aikawa* et. al. proposed a design method based on the SP method that it can handle filters with such characteristics. However, to obtain the minimum phase FIR filters, this method also needs factorization as same as HS method. Therefore, it is difficult to obtain the transfer function of minimum phase FIR filters with high order. In [4], *Okuda* et al. proposed an algorithm that based on solving a least squares problem iteratively. However, this method also needs factorization, though the accuracy is not worse than the HS method. And, the design method of minimum phase FIR filters with complex coefficient has not been proposed until now.

In this paper, we present a directly design method of minimum phase FIR filter with complex coefficients using the SP method. In general, it is necessary to restrict the zeros of transfer function inside the unit circle or on the unit circle to directly obtain the transfer function of minimum phase FIR filters. The design problem of the transfer function restricted the zeros is complex approximation problem. We use a modified SP method [7] to solve a complex approximation problem. The filter obtained by this method is optimal in the Chebyshev sense because we use the transformation which completely converts a complex approximation problem into a real approximation problem. Moreover, we use the Roushe's theorem for coefficients update algorithm in the modified SP method to restrict the zeros of transfer function inside the unit circle or on the unit circle. Therefore, the proposed method is possible to design the transfer function of high order further than conventional method, since it is possible to directly obtain the transfer function of minimum phase FIR filter without using factorization. Moreover, it can also design filters with more than tow different gain level in each stopband. Finally, the usefulness of the proposed method is verified using examples.

2. Design Problem

The frequency response of the FIR filters is expressed by

$$H(\omega) = \sum_{i=0}^{N} a_i e^{-ji\omega}, \tag{1}$$

where $a_i; i = 0,1,\cdots,N$ are complex filter coefficients.

Then, our design problem is to find the optimal filter coefficients minimizing the complex error

$$|\gamma(\omega)| = \left| D(\omega) - \sum_{i=0}^{N} a_i e^{-ji\omega} \right| \le \lambda(\omega), \tag{2}$$

where $D(\omega)$ is the desired amplitude response and $\lambda(\omega)$ is the positive maximum allowable deviation from the desired frequency response.

Since $\gamma(\omega)$ in (2) is a complex function, we use the expanded SP method to solve the complex approximation problem. In this method [7], by using a simple transformation, (2) can be converted to a linear optimization problem in the real domain as

$$\begin{aligned}|\gamma(\omega)| = \Big| D(\omega) - \sum_{i=0}^{N} \mathrm{Re}(a_i)\cos(-i\omega + 2\pi t) \\ + \sum_{i=0}^{N} \mathrm{Im}(a_i)\sin(-i\omega + 2\pi t)\Big| \le \lambda(\omega)\end{aligned} . \quad (3)$$

Where t is a rotation parameter and is defined as

$$t = -(0.5/\pi)\tan^{-1}(y/x) . \quad (4)$$

Where x and y to decide t are

$$x = -\sum_{i=0}^{N} \mathrm{Re}(a_i)\cos(-i\omega) + \sum_{i=0}^{N} \mathrm{Im}(a_i)\sin(-i\omega) \quad (5)$$

and

$$y = -\sum_{i=0}^{N} \mathrm{Re}(a_i)\sin(-i\omega) - \sum_{i=0}^{N} \mathrm{Im}(a_i)\cos(-i\omega) , \quad (6)$$

respectively.

3. Design by Successive Projections Method

SP method is one of the iterative techniques. Thus, the error function in nth iteration step is defined as

$$\begin{aligned}e(\omega, \mathbf{a}^n, t) = \Big| D(\omega) - \sum_{i=0}^{N} \mathrm{Re}(a_i^n)\cos(-i\omega + 2\pi t) \\ + \sum_{i=0}^{N-1} \mathrm{Im}(a_i^n)\sin(-i\omega + 2\pi t)\Big|\end{aligned} , \quad (7)$$

where $\mathbf{a} = [a_0, a_1, \cdots, a_N]$.

The design problem of filter by SP method is to minimize of the function F given by

$$F = \left\| \mathbf{a}^{n+1} - \mathbf{a}^n \right\| + \alpha\left[e\left(\omega_M, \mathbf{a}^{n+1}, t_M\right) - \lambda\left(\omega_M\right) \right] , \quad (8)$$

where α is the Lagrange multiplier, and $\mathbf{a}^{n+1}$ is the projection of $\mathbf{a}^n$ on the following N-dimentional Euclidean real space,

$$\phi_{n+1} = \{ h : e(\omega_M, \mathbf{a}^{n+1}, t_M) \le \lambda(\omega_M) \} \quad (9)$$

the set associated with (ω_M, t_M). Moreover,

$$\left\| \mathbf{a}^{n+1} - \mathbf{a}^n \right\| = \sum_{i=0}^{M} (a_i^{n+1} - a_i^n)^2 . \quad (10)$$

Hence, the design problem is to minimize the objective function (10) under constraint condition:

$$e(\omega_M, \mathbf{a}^{n+1}, t_M) - \lambda(\omega_M) = 0 \quad (11)$$

in N-dimentional Euclidean real space.

(1) *In Case of Complex Coefficients*

The evaluation function F of (8) becomes minimum value in satisfying (12) and (13) as follows.

$$\begin{aligned}\frac{\partial F}{\partial \mathrm{Re}(a_i^{n+1})} = 2\left(\mathrm{Re}(a_i^{n+1}) - \mathrm{Re}(a_i^n) \right) \\ - \alpha\cos(-i\omega_M + 2\pi t_M) = 0\end{aligned} . \quad (12)$$

$$\begin{aligned}\frac{\partial F}{\partial \mathrm{Im}(a_i^{n+1})} = 2\left(\mathrm{Im}(a_i^{n+1}) - \mathrm{Im}(a_i^n) \right) \\ + \alpha\sin(-i\omega_M + 2\pi t_M) = 0\end{aligned} . \quad (13)$$

Therefore, we can obtain

$$\mathrm{Re}\left\{ a_i^{n+1} \right\} = \mathrm{Re}\left\{ a_i^n \right\} + \frac{\alpha}{2}\cos(-i\omega_M + 2\pi t_M) \quad (14)$$

and

$$\mathrm{Im}\left\{ a_i^{n+1} \right\} = \mathrm{Im}\left\{ a_i^n \right\} - \frac{\alpha}{2}\sin(-i\omega_M + 2\pi t_M) . \quad (15)$$

Substituting (14) and (15) in the constraint condition (11), we can get

$$\alpha = \frac{2\left\{ e(\omega_M, \mathbf{a}^n, t_M) - \lambda(\omega_M) \right\}}{N+1} . \quad (16)$$

Furthermore, substituting (16) in (14) and (15), real part and complex part of the equation of updating coefficients in n+1th iteration step are obtained

$$\begin{aligned}\mathrm{Re}\left(a_i^{n+1}\right) = \mathrm{Re}\left(a_i^n\right) \\ + \frac{e\left(\omega_M, \mathbf{a}^n, t_M\right) - \lambda\left(\omega_M\right)}{N+1}\cos(-i\omega_M + 2\pi t_M)\end{aligned} \quad (17)$$

and

$$\begin{aligned}\mathrm{Im}\left(a_i^{n+1}\right) = \mathrm{Im}\left(a_i^n\right) \\ - \frac{e\left(\omega_M, \mathbf{a}^n, t_M\right) - \lambda\left(\omega_M\right)}{N+1}\sin(-i\omega_M + 2\pi t_M)\end{aligned} , \quad (18)$$

respectively. Therefore, the equation of updating coefficient of complex coefficient a_i is formulated as

$$a_i^{n+1} = \begin{cases} a_i^n + \delta_i^{n+1} & \cdots \text{if } e(\omega_M, \mathbf{a}^n, t_M) > \lambda(\omega_M) \\ a_i^n & \cdots \text{if } e(\omega_M, \mathbf{a}^n, t_M) \le \lambda(\omega_M) \end{cases} , \quad (19)$$

where

$$\delta_i^{n+1} = \frac{e\left(\omega_M, \mathbf{a}^n, t_M\right) - \lambda\left(\omega_M\right)}{N+1} \exp - j(-i\omega_M + 2\pi t_M) . \quad (20)$$

(2) *In Case of Real Coefficients*

In case of real coefficients, filter coefficients are $h_i = \mathrm{Re}\{a_i\}$ because $\mathrm{Im}\{a_i\} = 0$. Therefore, the Lagrange multiplier becomes

$$\alpha = \frac{2\left\{ e\left(\omega_M, \boldsymbol{h}^n, t_M\right) - \lambda\left(\omega_M\right) \right\}}{\sum_{i=0}^{N} \{\cos(-i\omega_M + 2\pi t_M)\}^2} , \quad (21)$$

when only real part of (12) and (13) is considered. Substituting (21) in (14), the equation of updating coefficient of real coefficient h_i is formulated as

$$h_i^{n+1} = \begin{cases} h_i^n + \rho_i^{n+1} & \cdots \text{if } e(\omega_M, \boldsymbol{h}^n, t_M) > \lambda(\omega_M) \\ h_i^n & \cdots \text{if } e(\omega_M, \boldsymbol{h}^n, t_M) \le \lambda(\omega_M) \end{cases} , \quad (22)$$

where

$$\rho_i^{n+1} = \frac{e\left(\omega_M, \boldsymbol{h}^n, t_M\right) - \lambda\left(\omega_M\right)}{\sum_{i=0}^{N} \{\cos(-i\omega_M + 2\pi t_M)\}^2}\cos(-i\omega_M + 2\pi t_M) . \quad (23)$$

By the way, all zeros of the transfer function of the minimum phase FIR filter must exist inside the unit circle in the passband and on the unit circle in the stopband. However, (19) and (22) are not guaranteed that zeros after updating coefficient are located inside the unit circle or on the unit circle. Therefore, the obtained filter by this iterative algorithm is not guaranteed to be minimum phase FIR filter.

4. The Expansion of the Updating Coefficients by Roushe's Theorem

In order to restrict zeros inside the unit circle in the passband and on the unit circle in the stopband, we expand updating coefficients of the SP method using the Rouche's theorem.

[Rouche's Theorem]

If $f(z)$ and $g(z)$ are analytic inside and on a closed contour *C*, and $|f(z)| > |g(z)|$ on *C*, then $f(z)$ and $f(z)+g(z)$ is have the same number of zeros inside *C*.

A proof of this theorem can be found in [8].

Let $f(z) = z^N H(z)$ and $g(z) = z^N \Delta(z)$ to use Roushe's theorem, where $H(z)$ is the actual coefficient polynomial of the desired transfer function and $\Delta(z)$ is an update value added to the actual coefficient polynomial $H(z)$ in each iteration step. The function $f(z)$ and $g(z)$ are analytic everywhere except at $z = \infty$. And, we choose C to be a unit circle centered at origin of the complex domain. If $H(z)$ has all its zeros inside this circle, then the new coefficient polynomial $H(z) + \beta\Delta(z)$, $(0 < \beta < 1)$, will still have all its zeros inside this circle if the update $\beta\Delta(z)$ satisfies $|\beta\Delta(z)| \le |H(z)|$ on the unit circle.

(1) *In Case of Complex Coefficients*

The equation of updating coefficients is modified as follows.

$$a_i^{n+1} = \begin{cases} a_i^n + \beta\delta_i^{n+1} & \cdots \text{if } e(\omega_M, \mathbf{a}^n, t_M) > \lambda(\omega_M) \\ a_i^n & \cdots \text{if } e(\omega_M, \mathbf{a}^n, t_M) \le \lambda(\omega_M) \end{cases}, \tag{24}$$

where δ_i^{n+1} is eq. (20) and β is the weight parameter as all zeros exist inside the unit circle and on the unit circle. Because $\Delta(z)$ is Z-transform of δ_i , in iteration step *n*+1 the constraint on δ_i^{n+1} can be formulated as

$$\left|\beta \sum_{i=1}^{N} \delta_i^{n+1} z^{-i}\right| - \left|H^n(z)\right| \le 0\,. \tag{25}$$

We choose a maximum β satisfying (25).

(2) *In Case of Real Coefficients*

The equation of updating coefficients is modified as follows.

$$h_i^{n+1} = \begin{cases} h_i^n + \beta\rho_i^{n+1} & \cdots \text{if } e(\omega_M, \boldsymbol{h}^n, t_M) > \lambda(\omega_M) \\ h_i^n & \cdots \text{if } e(\omega_M, \boldsymbol{h}^n, t_M) \le \lambda(\omega_M) \end{cases}, \tag{26}$$

where ρ_i^{n+1} is eq. (23) and β is the weight parameter as all zeros exist inside the unit circle and on the unit circle. Because $\Delta(z)$ is Z-transform of ρ_i , in iteration step *n*+1 the constraint on ρ_i^{n+1} can be formulated as

$$\left|\beta \sum_{i=1}^{N} \rho_i^{n+1} z^{-i}\right| - \left|H^n(z)\right| \le 0\,. \tag{27}$$

We choose a maximum β satisfying (27).

5. Design Examples

Example. 1

We design FIR filters with complex coefficients of specification shown as following.

$$D(\omega) = \begin{cases} 1 & \cdots 0.04 \le \omega \le 0.4\pi \\ 0 & \cdots -\pi < \omega \le -0.04\pi, 0.48\pi \le \omega \le \pi \end{cases}$$

$$\lambda(\omega) = \begin{cases} 0.0400 & \cdots 0.04\pi \le \omega \le 0.4\pi \\ 0.0062 & \cdots -\pi < \omega \le -0.04\pi, 0.48\pi \le \omega \le \pi \end{cases}$$

We choose $N = 36$. The initial coefficient of transfer function is chosen zeros to be in the unit circle. The magnitude response of the proposed filter is shown in Fig.1. It is clear from Fig.1 that we obtained equiripple characteristics of magnitude response. Therefore, the obtained filter is optimal in the Chebyshev sense. And, it is proven that the complex coefficients are required, since it is the asymmetry for the origin. The zeros location of the resulting filter is shown in Fig.2. It is clear from Fig.2 that the zeros of the obtained filter exist inside unit circle in the passband and on the unit circle in the stopband. Therefore, the obtained filter has the minimum phase characteristics.

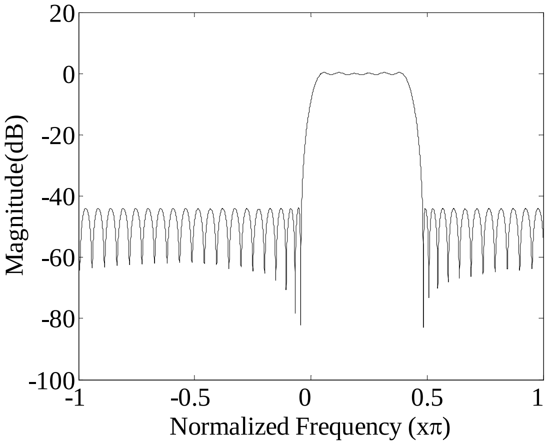

Fig.1. The amplitude response of FIR filter with complex coefficients

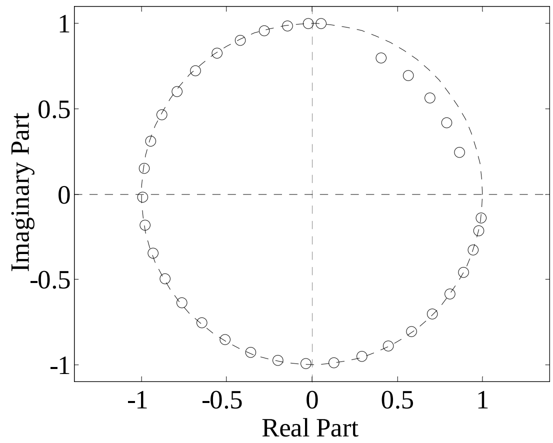

Fig.2. Zero locations of FIR filter with complex coefficients

Example. 2

Next, we design FIR filters of specification shown as following.

$$D(\omega) = \begin{cases} 1 & \cdots 0 \le \omega \le 0.1\pi \\ 0 & \cdots 0.13\pi \le \omega \le \pi \end{cases}$$

$$\lambda(\omega)=\begin{cases}0.0023 & \cdots 0\le\omega\le 0.1\pi\\ 0.0022 & \cdots 0.13\le\omega\le\pi\end{cases}$$

We choose $N=150$. The initial coefficient of transfer function is chosen zeros to be in the unit circle. The 300th order linear phase filter must be designed, if the filter of this specification is designed using the conventional method [1]-[4]. However, the 150th order filter can be directly designed in our proposed method, because the factorization is not required. The magnitude response of the proposed filter is shown in Fig.3. It is clear from Fig.3 that we obtained equiripple characteristics of magnitude response. Therefore, the obtained filter is optimal in the Chebyshev sense. The zeros location of the resulting filter is shown in Fig.4. It is clear from Fig.4 that the zeros of the obtained filter exist inside unit circle in the passband and on the unit circle in the stopband. Therefore, the obtained filter has the minimum phase characteristics.

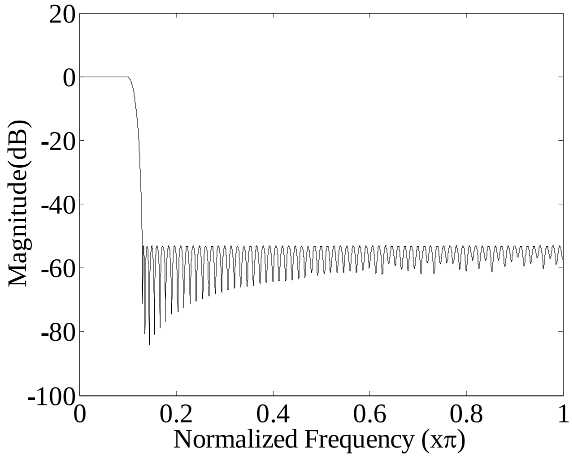

Fig.3. The amplitude response of FIR filter with real coefficients

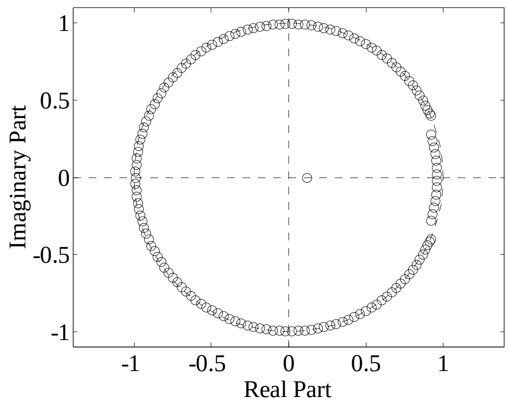

Fig.4. Zero locations of FIR filter with real coefficients

6. Conclusion

In this paper, we presented a directly design method of minimum phase FIR filters with complex coefficients using the SP method. In order to directly approximate the transfer function of FIR filter with the minimum phase characteristics generally, the approximation problem of the complex domain must be solved and the zeros of the transfer function is limited inside the unit circle and on the unit circle. To solve a complex approximation problem, we use a modified SP method. The filter obtained by this method is optimum in the Chebyshev sense because we use the transformation which completely converts a complex approximation problem into a real approximation problem. Moreover, we use Roushe's theorem for coefficients update algorithm in SP method to restrict the zeros of transfer function inside the unit circle and on the unit circle. Therefore, the transfer function of minimum phase FIR filter is obtained without using factorization. Finally, the usefulness of the proposed method is verified using examples.

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