ON ROBUST CAPON BEAMFORMING AND DIAGONAL LOADING

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ABSTRACT

Whenever the knowledge of the array steering vector is imprecise (as is often the case in practice), the performance of the Capon beamformer may become worse than that of the standard beamformer. Diagonal loading (including its extended versions) has been a popular approach to improve the robustness of the Capon beamformer. In this paper we show that a natural extension of the Capon beamformer to the case of uncertain steering vectors also belongs to the class of diagonal loading approaches but the amount of diagonal loading can be precisely calculated based on the uncertainty set of the steering vector. The proposed robust Capon beamformer can be efficiently computed at a comparable cost with that of the standard Capon beamformer. Its excellent performance is demonstrated via a number of numerical examples.

1. INTRODUCTION

The Capon beamformer has better resolution and much better interference rejection capability than the standard (data-independent) beamformer, provided that the array steering vector corresponding to the signal of interest (SOI) is accurately known. However, in practical applications, the knowledge of the SOI steering vector is often imprecise due to the differences between the assumed signal arrival angle and the true arrival angle or between the assumed array response and the true array response (array calibration errors). Whenever this happens, the performance of the Capon beamformer may become worse than that of the standard beamformers. To account for the array steering vector errors, additional linear constraints, including point and derivative constraints, can be imposed to improve the robustness of the Capon beamformer. However, these constraints are not explicitly related to the uncertainty of the array steering vector. Moreover, for every additional linear constraint imposed, the beamformer loses one degree of freedom (DOF) for interference suppression. Diagonal loading (including its extended versions) has been a popular approach to improve the robustness of the Capon beamformer. However, for most of these methods, it is not clear how to choose the diagonal loading level based on the uncertainty of the array steering vector.

Recently some methods with a clear theoretical background have been proposed, see e.g., [1, 2, 3, 4], which, unlike the early methods, make explicit use of an uncertainty set of the array steering vector. In [3], a polyhedron is used to describe the uncertainty set, whereas spherical and ellipsoidal (including flat ellipsoidal) uncertainty sets are considered in [1, 2, 4]. The robust Capon beamforming approaches presented in [1, 2] couple the formulation of the standard Capon beamformer (SCB) in [5] with a spherical or ellipsoidal uncertainty set of the array steering vector. Our RCB approach is different from those in [1, 2] in that we couple the formulation of SCB in [6] with an ellipsoidal uncertainty set. In addition, our RCB gives a simple way of eliminating the scaling ambiguity when estimating the power of the desired signal while the approaches in [1, 2] did not consider the scaling ambiguity problem.

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In this paper we show how to efficiently compute our robust Capon beamformer by using the Lagrange multiplier methodology. It turns out that our RCB also belongs to the class of diagonal loading approaches and that the amount of diagonal loading can be precisely calculated based on the ellipsoidal uncertainty set of the array steering vector. Numerical examples are presented to demonstrate the effectiveness of our RCB for SOI power estimation.

2. PROBLEM FORMULATION

Consider an array comprising M sensors and let $\mathbf{R}$ denote the theoretical covariance matrix of the array output vector. We assume that $\mathbf{R} > 0$ (positive definite) has the following form:

$$\mathbf{R} = \sigma_0^2 \mathbf{a}_0 \mathbf{a}_0^* + \sum_{k=1}^{K} \sigma_k^2 \mathbf{a}_k \mathbf{a}_k^* + \mathbf{Q} \tag{1}$$

where $\left(\sigma_0^2, \{\sigma_k^2\}_{k=1}^K\right)$ are the powers of the $(K+1)$ uncorrelated signals impinging on the array, $\left(\mathbf{a}_0, \{\mathbf{a}_k\}_{k=1}^K\right)$ are the so-called steering vectors that are functions of the location parameters of the sources emitting the signals (e.g., their directions of arrival (DOAs)), $(\cdot)^*$ denotes the conjugate transpose, and $\mathbf{Q}$ is the noise covariance matrix. In what follows we assume that the first term in (1) corresponds to the SOI and the remaining rank-one terms $\{\sigma_k^2 \mathbf{a}_k \mathbf{a}_k^*\}_{k=1}^K$ to K interferences. To avoid ambiguities, we assume that

$$\|\mathbf{a}_0\|^2 = M \tag{2}$$

where $\|\cdot\|$ denotes the Euclidean norm. In practical applications, $\mathbf{R}$ is replaced by the sample covariance matrix $\hat{\mathbf{R}}$, where

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}_n \mathbf{x}_n^* \tag{3}$$

with N denoting the number of snapshots and $\mathbf{x}_n$ representing the nth snapshot.

The *robust beamforming problem* we will deal with in this paper can now be briefly stated as follows: extend the Capon beamformer so as to be able to accurately determine the power of SOI even when only an imprecise knowledge of its steering vector, $\mathbf{a}_0$, is available. More specifically, we assume that the only knowledge we have about $\mathbf{a}_0$ is that it belongs to the following uncertainty ellipsoid (see Section 4 for the flat ellipsoid case):

$$[\mathbf{a}_0 - \bar{\mathbf{a}}]^* \mathbf{C}^{-1} [\mathbf{a}_0 - \bar{\mathbf{a}}] \leq 1 \quad (4)$$

where $\bar{\mathbf{a}}$ and $\mathbf{C}$ (a positive definite matrix) are given.

3. ROBUST CAPON BEAMFORMING

The common formulation of the beamforming problem that leads to the SCB is as follows (see, e.g., [5, 7]).

(a) Determine the $M \times 1$ vector $\mathbf{w}_0$ that is the solution to the following linearly constrained quadratic problem:

$$\min_{\mathbf{w}} \mathbf{w}^* \mathbf{R} \mathbf{w} \quad \text{subject to} \quad \mathbf{w}^* \mathbf{a}_0 = 1 \quad (5)$$

(b) Use $\mathbf{w}_0^* \mathbf{R} \mathbf{w}_0$ as an estimate of σ_0^2.

The solution to (5) is easily derived:

$$\mathbf{w}_0 = \frac{\mathbf{R}^{-1}\mathbf{a}_0}{\mathbf{a}_0^* \mathbf{R}^{-1} \mathbf{a}_0} \quad (6)$$

Using (6) in Step (b) above yields the following estimate of σ_0^2:

$$\tilde{\sigma}_0^2 = \frac{1}{\mathbf{a}_0^* \mathbf{R}^{-1} \mathbf{a}_0} \quad (7)$$

To derive our robust Capon beamforming approach, we use the reformulation of the Capon beamforming problem in [6] (also see [4]) that we motivate below in a simple way to which we append the uncertainty set in (4). Proceeding in this way we *directly* obtain a robust estimate of σ_0^2, without any intermediate calculation of a vector $\mathbf{w}$ [4]:

$$\max_{\sigma^2, \mathbf{a}} \sigma^2 \quad \text{subject to}$$
$$\mathbf{R} - \sigma^2 \mathbf{a}\mathbf{a}^* \geq 0$$
$$(\mathbf{a} - \bar{\mathbf{a}})^* \mathbf{C}^{-1} (\mathbf{a} - \bar{\mathbf{a}}) \leq 1 \quad (8)$$

(where $\bar{\mathbf{a}}$ and $\mathbf{C}$ are given).

For any given $\mathbf{a}$, the solution $\hat{\sigma}_0^2$ to (8) is indeed given by the counterpart of (7) (see [4]). Hence (8) can be reduced to the following problem

$$\min_{\mathbf{a}} \mathbf{a}^* \mathbf{R}^{-1} \mathbf{a} \quad \text{subject to} \quad (\mathbf{a} - \bar{\mathbf{a}})^* \mathbf{C}^{-1} (\mathbf{a} - \bar{\mathbf{a}}) \leq 1 \quad (9)$$

Without loss of generality, we will consider solving (9) for $\mathbf{C} = \epsilon \mathbf{I}$, i.e., solving the following quadratic optimization problem under a spherical constraint:

$$\min_{\mathbf{a}} \mathbf{a}^* \mathbf{R}^{-1} \mathbf{a} \quad \text{subject to} \quad \|\mathbf{a} - \bar{\mathbf{a}}\|^2 \leq \epsilon \quad (10)$$

Note that if $\mathbf{C}$ is not a scaled identity matrix, we can convert the problem into the same form as (10).

To exclude the trivial solution $\mathbf{a} = \mathbf{0}$ to (10), we assume that

$$\|\bar{\mathbf{a}}\|^2 > \epsilon \quad (11)$$

Because the solution to (10) (under (11)) will evidently occur on the boundary of the constraint set, we can re-formulate (10) as the following quadratic problem with a quadratic equality constraint:

$$\min_{\mathbf{a}} \mathbf{a}^* \mathbf{R}^{-1} \mathbf{a} \quad \text{subject to} \quad \|\mathbf{a} - \bar{\mathbf{a}}\|^2 = \epsilon \quad (12)$$

This problem can be solved by using the *Lagrange multiplier methodology*, which is based on the function:

$$f = \mathbf{a}^* \mathbf{R}^{-1} \mathbf{a} + \lambda \left(\|\mathbf{a} - \bar{\mathbf{a}}\|^2 - \epsilon \right) \quad (13)$$

where $\lambda \geq 0$ is the Lagrange multiplier [8]. Differentiation of (13) with respect to $\mathbf{a}$ gives the optimal solution $\hat{\mathbf{a}}_0$:

$$\mathbf{R}^{-1} \hat{\mathbf{a}}_0 + \lambda (\hat{\mathbf{a}}_0 - \bar{\mathbf{a}}) = 0 \quad (14)$$

The above equation yields

$$\hat{\mathbf{a}}_0 = \left(\frac{\mathbf{R}^{-1}}{\lambda} + \mathbf{I} \right)^{-1} \bar{\mathbf{a}} \quad (15)$$
$$= \bar{\mathbf{a}} - (\mathbf{I} + \lambda \mathbf{R})^{-1} \bar{\mathbf{a}} \quad (16)$$

where we have used the matrix inversion lemma to obtain the second equality. The Lagrange multiplier $\lambda \geq 0$ is obtained as the solution to the constraint equation:

$$g(\lambda) \triangleq \left\| (\mathbf{I} + \lambda \mathbf{R})^{-1} \bar{\mathbf{a}} \right\|^2 = \epsilon \quad (17)$$

Let

$$\mathbf{R} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^* \quad (18)$$

where the columns of $\mathbf{U}$ contain the eigenvectors of $\mathbf{R}$ and the diagonal elements of the diagonal matrix $\mathbf{\Lambda}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$, are the corresponding eigenvalues. Let

$$\mathbf{z} = \mathbf{U}^* \bar{\mathbf{a}} \quad (19)$$

and let z_m denote the mth element of $\mathbf{z}$. Then (17) can be written as

$$g(\lambda) = \sum_{m=1}^{M} \frac{|z_m|^2}{(1 + \lambda \lambda_m)^2} = \epsilon \quad (20)$$

Note that $g(\lambda)$ is a monotonically decreasing function of $\lambda \geq 0$. We can use, e.g., Newton's method, to determine the Lagrange multiplier λ from (20), and $\hat{\mathbf{a}}_0$ is then determined by using (16) and $\hat{\sigma}_0^2$ by using (7) with $\mathbf{a}_0$ replaced by $\hat{\mathbf{a}}_0$. Hence the major computational demand of our RCB comes from the eigendecomposition of the Hermitian matrix $\mathbf{R}$, which requires $O(M^3)$ flops. Therefore, the computational complexity of our RCB is comparable to that of the SCB.

Next observe that both the power and the steering vector of SOI are treated as unknowns in our robust Capon beamforming formulation (see (8)), and hence that there is a "scaling ambiguity" in the SOI covariance term in the sense that $(\sigma^2, \mathbf{a})$ and $(\sigma^2/\alpha, \alpha^{1/2}\mathbf{a})$ (for any $\alpha > 0$) give the same term $\sigma^2 \mathbf{a}\mathbf{a}^*$. To eliminate this ambiguity, we use the knowledge that $\|\mathbf{a}_0\|^2 = M$ (see (2)) and hence estimate σ_0^2 as [4]

$$\hat{\hat{\sigma}}_0^2 = \hat{\sigma}_0^2 \|\hat{\mathbf{a}}_0\|^2 / M \quad (21)$$

The numerical examples in [4] confirm that $\hat{\hat{\sigma}}_0^2$ is a (much) more accurate estimate of σ_0^2 than $\hat{\sigma}_0^2$.

Unlike our approach, the approaches of [1] and [2] do not provide any *direct* estimate $\hat{\mathbf{a}}_0$. Hence they do not dispose of a simple way (such as (21)) to eliminate the scaling ambiguity of the SOI power estimation that is likely a problem for all robust beamforming approaches (this problem was in fact ignored in both [1] and [2]). Note that SOI power estimation is the main goal in many applications including radar, sonar, and acoustic imaging.

In other applications, such as communications, the focus is on SOI waveform estimation. Let $s_0(n)$ denote the waveform of the SOI. Then once we have estimated the SOI steering vector with our RCB, $s_0(n)$ can be estimated like in the SCB as follows:

$$\hat{s}_0(n) = \hat{\mathbf{w}}_0^* \mathbf{x}_n \tag{22}$$

where $\hat{\mathbf{a}}_0$ in (15) is used to replace $\mathbf{a}_0$ in (6) to obtain $\hat{\mathbf{w}}_0$:

$$\hat{\mathbf{w}}_0 = \frac{\mathbf{R}^{-1}\hat{\mathbf{a}}_0}{\hat{\mathbf{a}}_0^*\mathbf{R}^{-1}\hat{\mathbf{a}}_0} \tag{23}$$

$$= \frac{\left(\mathbf{R}+\frac{1}{\lambda}\mathbf{I}\right)^{-1}\bar{\mathbf{a}}}{\bar{\mathbf{a}}^*\left(\mathbf{R}+\frac{1}{\lambda}\mathbf{I}\right)^{-1}\mathbf{R}\left(\mathbf{R}+\frac{1}{\lambda}\mathbf{I}\right)^{-1}\bar{\mathbf{a}}} \tag{24}$$

Note that our robust Capon weight vector has the form of diagonal loading except for the real-valued scaling factor in the denominator of (24). However, the scaling factor is not really important since the quality of the SOI waveform estimate is typically expressed by the signal-to-interference-plus-noise ratio (SINR)

$$\text{SINR} = \frac{\sigma_0^2|\hat{\mathbf{w}}_0^*\mathbf{a}_0|^2}{\hat{\mathbf{w}}_0^*\left(\sum_{k=1}^K \sigma_k^2\mathbf{a}_k\mathbf{a}_k^* + \mathbf{Q}\right)\hat{\mathbf{w}}_0} \tag{25}$$

which is independent of the scaling of the weight vector.

We remark that the discussions above indicate that our robust Capon beamforming approach belongs to the class of (extended) diagonally loaded Capon beamforming approaches. However, unlike earlier approaches, our approach can be used to determine exactly the optimal amount of diagonal loading needed for a given ellipsoidal uncertainty set of the steering vector, at a very modest computational cost.

Our approach is different from the recent RCB approaches in [1, 2]. The latter approaches extended Step (a) of SCB to take into account the fact that when there is uncertainty in $\mathbf{a}_0$, the constraint on $\mathbf{w}^*\mathbf{a}_0$ in (6) should be replaced with a constraint on $\mathbf{w}^*\mathbf{a}$ for any vector $\mathbf{a}$ in the uncertainty set (the constraints on $\mathbf{w}^*\mathbf{a}$ used in [1] and [2] are different from one another); then the so-obtained $\mathbf{w}$ is used in $\mathbf{w}^*\mathbf{R}\mathbf{w}$ to derive an estimate of σ_0^2, as in Step (b) of SCB. Despite the apparent differences in formulation, we can prove that our RCB gives the same weight vector as the RCBs presented in [1, 2], yet our RCB is computationally more efficient. We can also show that, although this aspect was ignored in [1, 2], the RCBs presented in [1, 2] can also be modified to eliminate the scaling ambiguity problem that occurs when estimating the SOI power σ_0^2.

4. FLAT ELLIPSOIDAL UNCERTAINTY SET

When the uncertainty set for $\mathbf{a}$ is a flat ellipsoid, as is considered in [2, 9] to make the uncertainty set as tight as possible (assuming that the available a priori information allows that), (8) becomes

$$\max_{\sigma^2,\mathbf{a}} \sigma^2 \quad \text{subject to} \quad \mathbf{R} - \sigma^2\mathbf{a}\mathbf{a}^* \geq 0$$
$$\mathbf{a} = \mathbf{B}\mathbf{u} + \bar{\mathbf{a}}, \quad \|\mathbf{u}\| \leq 1 \tag{26}$$

where $\mathbf{B}$ is an $M \times L$ matrix $(L < M)$ with full column rank and $\mathbf{u}$ is an $L \times 1$ vector. (When $L = M$, (26) becomes (4) with $\mathbf{C} = \mathbf{B}\mathbf{B}^*$.) The RCB optimization problem in (26) can be reduced to (see (9)):

$$\min_{\mathbf{u}} (\mathbf{B}\mathbf{u}+\bar{\mathbf{a}})^*\mathbf{R}^{-1}(\mathbf{B}\mathbf{u}+\bar{\mathbf{a}}) \quad \text{subject to} \quad \|\mathbf{u}\| \leq 1 \tag{27}$$

Note that

$$(\mathbf{B}\mathbf{u}+\bar{\mathbf{a}})^*\mathbf{R}^{-1}(\mathbf{B}\mathbf{u}+\bar{\mathbf{a}}) =$$
$$\mathbf{u}^*\mathbf{B}^*\mathbf{R}^{-1}\mathbf{B}\mathbf{u} + \bar{\mathbf{a}}^*\mathbf{R}^{-1}\mathbf{B}\mathbf{u} + \mathbf{u}^*\mathbf{B}^*\mathbf{R}^{-1}\bar{\mathbf{a}} + \bar{\mathbf{a}}^*\mathbf{R}^{-1}\bar{\mathbf{a}} \tag{28}$$

Let

$$\check{\mathbf{R}} = \mathbf{B}^*\mathbf{R}^{-1}\mathbf{B} > 0 \tag{29}$$

and

$$\bar{\check{\mathbf{a}}} = \mathbf{B}^*\mathbf{R}^{-1}\bar{\mathbf{a}} \tag{30}$$

Using (28)-(30) in (27) gives

$$\min_{\mathbf{u}} \mathbf{u}^*\check{\mathbf{R}}\mathbf{u} + \bar{\check{\mathbf{a}}}^*\mathbf{u} + \mathbf{u}^*\bar{\check{\mathbf{a}}} \quad \text{subject to} \quad \|\mathbf{u}\| \leq 1 \tag{31}$$

The *Lagrange multiplier methodology* is based on the function [10]

$$\check{f} = \mathbf{u}^*\check{\mathbf{R}}\mathbf{u} + \bar{\check{\mathbf{a}}}^*\mathbf{u} + \mathbf{u}^*\bar{\check{\mathbf{a}}} + \check{\lambda}(\mathbf{u}^*\mathbf{u} - 1) \tag{32}$$

where $\check{\lambda}$ is the Lagrange multiplier [8]. Differentiation of (32) with respect to $\mathbf{u}$ gives

$$\check{\mathbf{R}}\hat{\mathbf{u}} + \bar{\check{\mathbf{a}}} + \check{\lambda}\hat{\mathbf{u}} = 0 \tag{33}$$

which yields

$$\hat{\mathbf{u}} = -(\check{\mathbf{R}} + \check{\lambda}\mathbf{I})^{-1}\bar{\check{\mathbf{a}}} \tag{34}$$

where $\check{\lambda} \geq 0$ can be determined similar to Section 3. Then $\check{\lambda}$ is used in (34) to obtain the $\hat{\mathbf{u}}$ that solves (31) and $\hat{\mathbf{u}}$ is next used to obtain the optimal solution to (26) as:

$$\hat{\mathbf{a}}_0 = \mathbf{B}\hat{\mathbf{u}} + \bar{\mathbf{a}} \tag{35}$$

Hence, under the flat ellipsoidal constraint the complexity of our RCB is also $O(M^3)$ flops, which is on the same order as for SCB and is mainly due to computing $\mathbf{R}^{-1}$ and the eigendecomposition of $\check{\mathbf{R}}$. If $L \ll M$, then the complexity is mainly due to computing $\mathbf{R}^{-1}$. Note that here we only need $O(L^3)$ flops to compute $\check{\lambda}$ while the approach in [2] requires $O(M^3)$ flops (and $L \leq M$). Our approach is also simpler from a conceptual standpoint than that of [2].

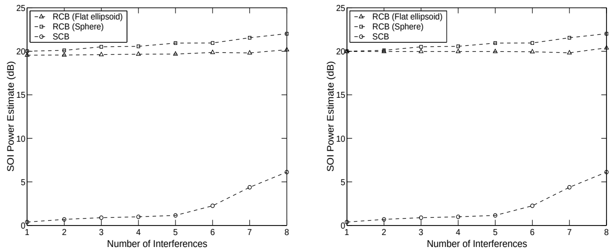
5. NUMERICAL EXAMPLES

Our main motivation for studying the RCB problem was an acoustic imaging application in which the goal was to estimate the SOI power in the presence of strong interferences as well as some uncertainty in the SOI steering vector. We assume a uniform linear array with $M = 10$ sensors and half-wavelength sensor spacing. We assume a spatially white Gaussian noise whose covariance matrix is given by $\mathbf{Q} = \mathbf{I}$. We assume that the steering vector uncertainty is due to the uncertainty in the SOI's direction of arrival θ_0, which we assume to be $\theta_0 + \Delta$. We assume that $\mathbf{a}(\theta_0)$ belongs to the spherical uncertainty set

$$\|\mathbf{a}(\theta_0) - \bar{\mathbf{a}}\|^2 \leq \epsilon; \quad \bar{\mathbf{a}} = \mathbf{a}(\theta_0 + \Delta) \tag{36}$$

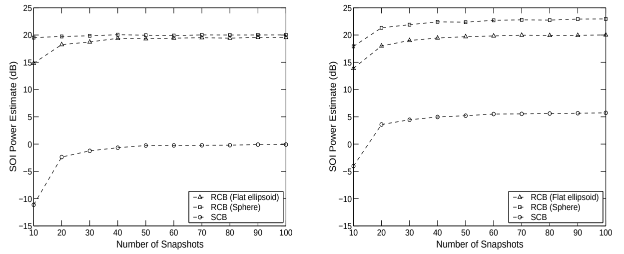
where ϵ is a user parameter. Let $\epsilon_0 = \|\mathbf{a}(\theta_0) - \bar{\mathbf{a}}\|^2$. Then choosing $\epsilon = \epsilon_0$ gives the smallest spherical set that includes $\mathbf{a}(\theta_0)$.

We examine the effects of the spherical and flat ellipsoidal constraints on SOI power estimation in the presence of several strong interferences. We will vary the number of interferences from $K = 1$ to $K = 8$. The power of SOI is $\sigma_0^2 = 20$ dB and the interference powers are $\sigma_1^2 = \cdots = \sigma_K^2 = 40$ dB. The SOI and interference directions of arrival are $\theta_0 = 10°, \theta_1 = -75°, \theta_2 = -60°, \theta_3 = -45°, \theta_4 = -30°, \theta_5 = -10°, \theta_6 = 25°, \theta_7 = 35°, \theta_8 = 50°$. We assume that there is a look direction mismatch corresponding to $\Delta = 2°$ and accordingly $\epsilon_0 = 3.1349$. Figure 1



(a) $\delta = 1.8°$ (b) $\delta = 2.4°$

Fig. 1. $\tilde{\sigma}_0^2$ (SCB), $\hat{\tilde{\sigma}}_0^2$ (RCB with flat ellipsoidal constraint with $L = 2$), and $\hat{\tilde{\sigma}}_0^2$ (RCB with spherical constraint), based on $\mathbf{R}$, versus the number of interferences K when (a) $\delta = 1.8°$ and (b) $\delta = 2.4°$. The true SOI power is 20 dB and $\epsilon_0 = 3.1349$ (corresponding to $\Delta = 2°$).



(a) $K = 1$ (b) $K = 8$

Fig. 2. Comparison of the SOI power estimates, versus N, obtained using SCB, RCB (with flat ellipsoidal constraint) and RCB (with spherical constraint), all with $\hat{\mathbf{R}}$, when $\delta = 2.4°$ for (a) $K = 1$ and (b) $K = 8$. The true SOI power is 20 dB and $\epsilon_0 = 3.1349$ (corresponding to $\Delta = 2°$).

shows the SOI power estimates, as a function of the number of interferences K, obtained by using SCB, RCB (with flat ellipsoidal constraint), and the more conservative RCB (with spherical constraint) all based on the theoretical array covariance matrix $\mathbf{R}$. For RCB with flat ellipsoidal constraint, we let $\mathbf{B}$ contain two columns with the first column being $\mathbf{a}(\theta_0+\Delta)-\mathbf{a}(\theta_0+\Delta-\delta)$ and the second column being $\mathbf{a}(\theta_0+\Delta)-\mathbf{a}(\theta_0+\Delta+\delta)$. Note that choosing $\delta = \Delta = 2°$ gives the smallest flat ellipsoid that this $\mathbf{B}$ can offer to include $\mathbf{a}(\theta_0)$. However, we do not know the exact look direction mismatch in practice. We choose $\delta = 1.8°$ and $\delta = 2.4°$ in Figures 1(a) and 1(b), respectively. For RCB with spherical constraint, we choose ϵ to be the larger of $\|\mathbf{a}(\theta_0 + \Delta) - \mathbf{a}(\theta_0 + \Delta - \delta)\|^2$ and $\|\mathbf{a}(\theta_0 + \Delta) - \mathbf{a}(\theta_0 + \Delta + \delta)\|^2$. Note that RCB with flat ellipsoidal constraint and RCB with spherical constraint perform similarly when K is small. However, the former is more accurate than the latter for large K. Figure 2 shows the SOI power estimates versus the number of snapshots N for $K = 1$ and $K = 8$ when the sample covariance matrix $\hat{\mathbf{R}}$ is used in lieu of the theoretical array covariance matrix in the beamformers and the average power estimates from 100 Monte-Carlo simulations are given. Note that for small K, RCB with spherical constraint converges faster than RCB with flat ellipsoidal constraint as N increases, while the latter converges faster than SCB. For large K, however, the convergence speeds of RCB with flat ellipsoidal constraint and RCB with spherical constraint are about the same as that of SCB; after convergence, the most accurate power estimate is provided by RCB with flat ellipsoidal constraint.

6. CONCLUSIONS

We have shown how to obtain a robust Capon beamformer (RCB) based on an ellipsoidal (including flat ellipsoidal) uncertainty set of the array steering vector, at a comparable computational cost with that associated with SCB. The data-adaptive RCB is much less sensitive to steering vector mismatches than the standard Capon beamformer (SCB) and yet it can retain the appealing properties of SCB including better resolution and much better interference rejection capability than the standard (data-independent) beamformer. We have shown that the RCB belongs to the class of diagonal loading approaches but the amount of diagonal loading can be precisely calculated based on the uncertainty set of the steering vector. The excellent performance of our RCB for SOI power estimation has been demonstrated via a number of numerical examples.

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