TIME-DELAY ESTIMATION IN MIXTURES

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ABSTRACT

We address the problem of passive blind estimation of time-delays for several mutually uncorrelated source signals received by a similar number of sensors. The mixtures at the receivers are modeled as unknown linear combinations of differently delayed versions of the source signals. The standard tools used in blind source separation (BSS) for either static or convolutive mixtures are inappropriate for this problem: The former is obviously under-parameterized, while the latter is over-parameterized and poorly suited for accommodating pure fractional delays. Thus, in this paper we propose a hybrid algorithm, which uses a specially parameterized approximate joint diagonalization of spectral matrices to estimate the delays (as well as the unknown mixing coefficients). The joint diagonalization algorithm is an extension of the iterative "AC-DC" algorithm, previously proposed in the context of BSS with static mixtures. We provide analytic expressions for all required steps for the two sensors / two sources case, and demonstrate the performance using simulations results.

1. INTRODUCTION

The problem of passive time-delay estimation has mainly been treated for the case of a single source signal, possibly in the presence of multipath replica thereof (see, e.g., [1]). When multiple (statistically independent) source signals are present, the problem becomes more closely related to that of blind source separation (BSS), especially when the use of prior knowledge (such as geometrical properties of the sensors array) is unfeasible or undesired.

The classical BSS problem has seen extensive treatment in the contexts of static and convolutive mixtures (e.g., [2, 3]).In this paper we address a hybrid model, in which the mixing is constrained to consist of pure delays in addition to static mixture coefficients. The delays are assumed to have occurred prior to the sampling process, and are therefore not necessarily an integer multiple of the sampling period. The (pre-sampled, continuous-time) source signals are assumed to be mutually uncorrelated, wide-sense stationary (WSS) with unknown spectra. They are further assumed to be band-limited, so that the sampling is at least at the Nyquist rate.

The problem of time-delay estimation for mixtures has seen little treatment in the literature so far. In [4] some preliminary analytic solutions are proposed, based on second and/or fourth order spectra; However, the model addressed does not allow for unknown mixing coefficients in addition to the delays (although it is indicated that one of the solutions can be adapted to accommodate ones). In [5] the case of fewer sensors than sources is addressed, but again there's no provision for unknown mixing coefficients. In [6] the delays are assumed to be integer multiples of the sampling period.

In this paper we address the following L sources - L sensors model (to be later reduced to $L = 2$):

$$x_p(t) = \sum_{q=1}^{L} a_{pq} s_q(t - \tau_{pq}) \quad p = 1, 2, \dots L \tag{1}$$

where $s_q(t)$ are zero-mean, WSS with unknown spectra, $x_p(t)$ are the observations, a_{pq} are the mixing coefficients and τ_{pq} are the delays from source q to sensor p. To mitigate the ambiguity associated with the sources' undetermined time-origin, we use as a "working assumption" zero delays from each source to the "respective" sensor, i.e., $\tau_{pp} = 0$ for $p = 1, 2, \dots L$.

The available data are samples of the continuous-time observations, $x_p[n] = x_p(nT) \quad n = 1, 2, \dots N$, where T is the sampling period (we shall use parentheses / brackets to enclose continuous- / discrete- time indices, respectively). It is assumed that all the source signals (and hence the observed signals) are bandlimited at (angular) frequency π/T. It is desired to estimate the relative delays τ_{pq} from the observed samples.

The paper is organized as follows: In the next section we present the estimation problem as a specially parameterized joint diagonalization problem in the frequency domain; In section 3 we propose an iterative algorithm for the joint diagonalization, based on an extension of the "AC-DC" algorithm [7]; In section 4 we present some simulations results.

2. FORMULATION AS A JOINT DIAGONALIZATION PROBLEM

The observations' correlation functions are given by

$$\begin{aligned} R^x_{mn}(\tau) &= E[x_m(t+\tau)x_n(t)] \\ &= \sum_{p=1}^{L}\sum_{q=1}^{L} a_{mp}a_{nq}E[s_p(t-\tau_{mp}+\tau)s_q(t-\tau_{nq})] \\ &= \sum_{q=1}^{L} a_{mq}a_{nq}R^s_q(\tau+\tau_{nq}-\tau_{mq}) \quad 1\le m,n\le L \end{aligned} \tag{2}$$

where $R^x_{mn}(\tau)$ denotes the correlation between the m-th and the n-th received signals, and $R^s_q(\tau)$ denotes the autocorrelation of the q-th source signal.

Fourier-transforming (2), we obtain

$$S^x_{mn}(\omega) = \sum_{q=1}^{L} a_{mq}a_{nq}S^s_q(\omega)e^{-j\omega(\tau_{mq}-\tau_{nq})} \qquad 1\le m,n\le L \tag{3}$$

where $S^x_{mn}(\omega)$ is the cross-spectrum between the m-th and n-th received signal and $S^s_q(\omega)$ is the q-th source's (unknown) spectrum. Eq. (3) can also be expressed in matrix-form as

$$\boldsymbol{S}_x(\omega) = \boldsymbol{B}(\omega)\boldsymbol{S}_s(\omega)\boldsymbol{B}^H(\omega) \tag{4}$$

where $\boldsymbol{S}_x(\omega)$ is an $L\times L$ matrix consisting of $S^x_{mn}(\omega)$ as the m,n-th element, $\boldsymbol{S}_s(\omega)$ is an $L\times L$ diagonal matrix consisting of $S^s_q(\omega)$ as its q,q-th elements, and $\boldsymbol{B}(\omega)$ is the $L\times L$ matrix given by $\boldsymbol{B}(\omega)=\boldsymbol{A}\odot\boldsymbol{D}(\omega)$, where $\odot$ denotes Hadamard's (element-wise) product, $\boldsymbol{A}$ is the constant matrix of mixing coefficients, whose m,n-th element is a_{mn}, and $\boldsymbol{D}(\omega)$ contains the delays, such that its m,n-th element is given by

$$D_{mn} = e^{-j\omega\tau_{mn}} \quad 1\le m,n\le L. \tag{5}$$

The cross-spectral matrices $\boldsymbol{S}_x(\omega)$ are unknown, but can be estimated from the available data, possibly by using the Discrete-Time Fourier Transform (DTFT) of a truncated series of unbiased cross-correlations estimates (Blackman-Tuckey's method, e.g., [8]). Specifically, to estimate the m,n-th element of $\boldsymbol{S}_x(\omega)$, we may use $\hat{S}^x_{mn}(\omega)=\sum_{l=-M}^{M}\hat{R}_{mn}[l]e^{-j\omega l}$, where

$$\hat{R}_{mn}[l] = \frac{1}{N-|l|}\sum_{p=1}^{N-|l|} x_m[p+l]x_n[p] \quad -M\le l\le M. \tag{6}$$

and M is the truncation-window length. If M is larger than the sum of the longest correlation length (among all source signals) and the maximal delay, then these are unbiased estimates of the desired (cross-) spectra.

Note that in the transition from continuous time to discrete time, the frequency axis is rescaled to the range $-\pi:\pi$, resulting in some constant (and irrelevant) scaling of the estimated spectra. Consequently, estimated delays will later have to be translated from sample units to time units via multiplication by T.

When estimated values, rather than true values, of $\boldsymbol{S}_x(\omega)$ are used, the equations (4) usually can no longer be satisfied simultaneously at all frequencies. Nevertheless, once $\boldsymbol{S}_x(\omega)$ is estimated at several frequencies $\omega_0,\omega_1,\dots\omega_K$, an estimate of the unknown parameters of interest can be obtained by resorting to approximate joint diagonalization (see e.g. [7, 3]), seeking to minimize the following least-squares (LS) criterion:

$$\min_{\boldsymbol{A},\boldsymbol{T},\boldsymbol{\Gamma}} C_{LS} \triangleq \sum_{k=0}^{K} ||\boldsymbol{S}_x(\omega_k)-\boldsymbol{B}(\omega_k)\boldsymbol{S}_s(\omega_k)\boldsymbol{B}^H(\omega_k)||_F^2 \tag{7}$$

where $\boldsymbol{T}$ is an $L\times L$ matrix containing the delay parameters τ_{mn}, $\boldsymbol{\Gamma}$ is an $L\times(K+1)$ matrix containing the sources' spectra, $\gamma_{mk}=S^s_m(\omega_k)\quad 1\le m\le L\quad 0\le k\le K$ and $||\cdot||_F^2$ denotes the squared Frobenius norm. Note that it is also possible to use a weighted LS criterion by introducing some positive weights w_k into the sum; however, to simplify the exposition, we shall not pursue this possibility in here.

Several algorithms exist for joint diagonalization of sets of matrices. However, these algorithms assume a fixed diagonalizing matrix $\boldsymbol{B}$, rather than $\boldsymbol{B}(\omega_k)$ which depends on the index k. In the next section we propose an extension of an existing joint diagonalization algorithm, namely the AC-DC algorithm [7], adapted to this minimization problem.

3. JOINT DIAGONALIZATION VIA THE EXTENDED AC-DC ALGORITHM

The AC-DC ("Alternating Columns / Diagonal Centers") algorithm [7] is an alternating-directions minimization algorithm, originally intended for the case of a fixed diagonalizing matrix $\boldsymbol{B}$. In our case the matrix $\boldsymbol{B}$ is not constant, but can be factored so as to depend on two constant matrices, $\boldsymbol{A}$ and $\boldsymbol{T}$. It is then possible to minimize w.r.t. each column of $\boldsymbol{A}$ and $\boldsymbol{T}$ separately, thus alternating between minimizations w.r.t.:

- $\boldsymbol{\Gamma}$ (in the DC phase);
- each column of $\boldsymbol{A}$ (in the AC-1 phase);
- each column of $\boldsymbol{T}$ (in the AC-2 phase).

3.1. The "DC" phase

In the DC phase we wish to minimize C_{LS} w.r.t. $\boldsymbol{\Gamma}$, with $\boldsymbol{A}$ and $\boldsymbol{T}$ fixed. Since the k-th column of $\boldsymbol{\Gamma}$ is the diagonal of $\boldsymbol{S}_s(\omega_k)$, it participates only in the k-th term of the sum in (7). Thus, the minimization can be decomposed into $K+1$ distinct minimization problems which are all linear in the unknown parameters, and thus admit the well-known linear LS solution. Specifically, note that each (k-th) term in the sum can be expressed as

$$||\boldsymbol{S}_x(\omega_k) - \boldsymbol{B}(\omega_k)\boldsymbol{S}_s(\omega_k)\boldsymbol{B}^H(\omega_k)||_F^2 = [\boldsymbol{y}_k - \boldsymbol{H}_k\boldsymbol{\gamma}_k]^H[\boldsymbol{y}_k - \boldsymbol{H}_k\boldsymbol{\gamma}_k] \quad (8)$$

where $\boldsymbol{\gamma}_k$ is the k-th column of $\boldsymbol{\Gamma}$, $\boldsymbol{y}_k \stackrel{\triangle}{=} \text{vec}\{\boldsymbol{S}_x(\omega_k)\}$ (vec$\{\cdot\}$ denoting the concatenation of the matrix' columns into one vector), and

$$\boldsymbol{H}_k = (\boldsymbol{B}(\omega_k)^* \otimes \mathbf{1}) \odot (\mathbf{1} \otimes \boldsymbol{B}(\omega_k)) \quad (9)$$

where $\mathbf{1}$ denotes an $L \times 1$ vector of 1-s, $\otimes$ denotes Kronecker's product, $\odot$ denotes Hadamard's (elementwise) product, and the superscript * denotes conjugation (note that this expression is sometimes referred to as the Khatri-Rao product of $\boldsymbol{B}^*$ and $\boldsymbol{B}$). The well-known minimizer of the linear LS problem is

$$\boldsymbol{\gamma}_k = [\boldsymbol{H}_k^H\boldsymbol{H}_k]^{-1}\boldsymbol{H}_k^H\boldsymbol{y}_k. \quad (10)$$

3.2. The "AC-1" phase

We now wish to minimize C_{LS} w.r.t. the l-th ($l = 1, 2, \ldots L$) column of $\boldsymbol{A}$, assuming the other columns, as well as $\boldsymbol{T}$ and $\boldsymbol{\Gamma}$, are fixed. Defining

$$\tilde{\boldsymbol{S}}(\omega_k) \stackrel{\triangle}{=} \boldsymbol{S}_x(\omega_k) - \sum_{\substack{n=1\\n\neq l}}^{L} S_n^s(\omega_k)\boldsymbol{b}_n(\omega_k)\boldsymbol{b}_n^H(\omega_k), \quad (11)$$

where $\boldsymbol{b}_n(\omega_k)$ is the n-th column of $\boldsymbol{B}(\omega_k)$, we can obtain, through some algebraic manipulations (using the fact that all $S_n^s(\omega_k)$ are real-valued, being the sources' spectrum)

$$C_{LS} = \tilde{C} - 2\sum_{k=0}^{K} S_l^s(\omega_k)\boldsymbol{b}_l^H(\omega_k)\tilde{\boldsymbol{S}}(\omega_k)\boldsymbol{b}_l(\omega_k) + \sum_{k=0}^{K}\left(\boldsymbol{b}_l^H(\omega_k)\boldsymbol{b}_l(\omega_k)\right)^2 S_l^{s2}(\omega_k) \quad (12)$$

where $\tilde{C}$ is an independent constant. Observe now, that $\boldsymbol{b}_l(\omega_k)$ can be written as

$$\boldsymbol{b}_l(\omega_k) = \boldsymbol{\Lambda}_l(\omega_k)\boldsymbol{a}_l \quad (13)$$

where $\boldsymbol{\Lambda}_l(\omega_k) = \text{diag}\{e^{-j\omega_k\tau_{1l}}, e^{-j\omega_k\tau_{2l}}, \ldots e^{-j\omega_k\tau_{Ll}}\}$. Consequently, C_{LS} can be further simplified,

$$C_{LS} = \tilde{C} - 2\boldsymbol{a}_l^T\left[\sum_{k=0}^{K} S_l^s(\omega_k)\boldsymbol{\Lambda}_l^H(\omega_k)\tilde{\boldsymbol{S}}(\omega_k)\boldsymbol{\Lambda}_l(\omega_k)\right]\boldsymbol{a}_l + (\boldsymbol{a}_l^T\boldsymbol{a}_l)^2\sum_{k=0}^{K} S_l^{s2}(\omega_k). \quad (14)$$

We can further decompose $\boldsymbol{a}_l$ into a scale a times a unit-norm vector $\boldsymbol{\alpha}$, thus reducing (14) into

$$C_{LS} = \tilde{C} - 2a^2\boldsymbol{\alpha}^T\boldsymbol{F}\boldsymbol{\alpha} + a^4 f \quad (15)$$

where $\boldsymbol{F}$ is the Hermitian matrix

$$\boldsymbol{F} \stackrel{\triangle}{=} \sum_{k=0}^{K} S_l^s(\omega_k)\boldsymbol{\Lambda}_l^H(\omega_k)\tilde{\boldsymbol{S}}(\omega_k)\boldsymbol{\Lambda}_l(\omega_k) \quad (16)$$

and $f = \sum_{k=0}^{K} S_l^{s2}(\omega_k)$. Differentiating (14) w.r.t. a and equating zero yields either the solution $a = 0$ or $a^2 = \boldsymbol{\alpha}^T\boldsymbol{F}\boldsymbol{\alpha}/f$, which, since $\boldsymbol{F}$ is Hermitian, is real-valued. Thus, if it is positive, then the minimizing a is its square root, otherwise it is zero. Consequently, if $\boldsymbol{F}$ is negative-definite, then minimization of C_{LS} w.r.t. $\boldsymbol{a}_l$ is attained by $\boldsymbol{a}_l = \mathbf{0}$. Normally, however, this is not the case, and substituting a^2 back into (15) reduces the problem into maximization w.r.t $\boldsymbol{\alpha}$ of $(\boldsymbol{\alpha}^T\boldsymbol{F}\boldsymbol{\alpha})^2$ subject to $\boldsymbol{\alpha}^T\boldsymbol{\alpha} = 1$. The desired solution is attained as the eigenvector of $\boldsymbol{F}$ associated with the largest (*positive*) eigenvalue.

3.3. The "AC-2" phase

It is now desired to minimize C_{LS} w.r.t. $\boldsymbol{\tau}_l$, the l-th ($l = 1, 2, \ldots L$) column of $\boldsymbol{T}$, assuming the other columns, as well as $\boldsymbol{A}$ and $\boldsymbol{\Gamma}$ are fixed. Since the dependence on the delays $\boldsymbol{\tau}_l$ appears only through $\boldsymbol{\Lambda}_l(\omega_k)$, it is evident from (14) that C_{LS} can be expressed as

$$C_{LS} = \tilde{\tilde{C}} - 2\boldsymbol{a}_l\left[\sum_{k=0}^{K}\boldsymbol{\Lambda}_l^H(\omega_k)\boldsymbol{G}(\omega_k)\boldsymbol{\Lambda}_l(\omega_k)\right]\boldsymbol{a}_l \quad (17)$$

$\tilde{\tilde{C}}$ being another constant, and $\boldsymbol{G}(\omega_k) \stackrel{\triangle}{=} S_l^s\tilde{\boldsymbol{S}}(\omega_k)$.

Differentiating w.r.t. τ_{pl} and equating zero, we obtain the following set of equations:

$$\frac{\partial C_{LS}}{\partial \tau_{pl}} = -2j\sum_{m=1}^{L} a_{pl}a_{ml}\cdot\sum_{k=0}^{K}\omega_k\Big(g_{pm}(\omega_k)e^{j\omega_k(\tau_{ml}-\tau_{pl})} - g_{mp}(\omega_k)e^{-j\omega_k(\tau_{ml}-\tau_{pl})}\Big) = 0 \quad 1 \leq p \leq L, p \neq l \quad (18)$$

where a_{ij} and $g_{ij}(\omega_k)$ denote the i,j-th elements of $\boldsymbol{A}$ and $\boldsymbol{G}(\omega_k)$, respectively. This set of equations is to be solved w.r.t. $\tau_{1l}, \tau_{2l}, \ldots \tau_{Ll}$ except for τ_{ll}, which by convention was set to zero.

We do not have an analytical solution to (18) for the general case. However, if we reduce the discussion to the case of $L = 2$ sensors and sources, and the frequencies $\{\omega_k\}_{k=0}^{K}$ are chosen as $\omega_k = k\Omega \quad k = 0, 1, \ldots K$ (with Ω a selected constant), then this set reduces to:

$$\sum_{k=0}^{K} k \left[g_{21}(k\Omega) e^{-j\Omega\tau_{21}k} - g_{12}(k\Omega) e^{j\Omega\tau_{21}k} \right] = 0 \quad \text{(19a)}$$

(for $l = 1$, $p = 2$), and

$$\sum_{k=0}^{K} k \left[g_{12}(k\Omega) e^{-j\Omega\tau_{12}k} - g_{21}(k\Omega) e^{j\Omega\tau_{12}k} \right] = 0 \quad \text{(19b)}$$

(for $l = 2$, $p = 1$). To proceed, we now define $\rho_{pl} \triangleq e^{j\Omega\tau_{pl}}$, so that (19a,19b) can be written as

$$\sum_{k=0}^{K} k[g_{pl}(k\Omega)\rho_{pl}^{-k} - g_{lp}(k\Omega)\rho_{pl}^{k}] = 0, \qquad (20)$$

which, after multiplication by ρ^K turns into a polynomial of degree $2K$ in ρ. Using polynomial rooting and selecting all unit-modulus roots $\hat{\rho}_{pl}$, yields all (possibly numerous) stationary points of $C_L S$ w.r.t. τ_{pl} via $\hat{\tau}_{pl} = \text{Imag}\{\log \hat{\rho}_{pl}\}/\Omega$. Each of these candidate solutions can be plugged into (17) for evaluation of C_{LS} in order to select the minimizing solution.

It is interesting to observe, that in the $L = 2$ case the dependence of (18) on $\boldsymbol{A}$ vanishes, so that this phase (AC-2) can be regarded an inseparable part of the previous phase (AC-1), since minimization w.r.t. both $\boldsymbol{a}_l$ and $\boldsymbol{\tau}_l$ can be attained simultaneously.

4. SIMULATIONS RESULTS

We compare the estimation accuracy to a method proposed by Comon and Emile in [4]. We used source signals generated as follows. Both signal were originally generated at a sample rate 10 times higher than the eventual processing sample rate, to enable fractional delays prior to "sampling". At the higher sample rate s_1 and s_2 were first generated as first-order Auto-Regressive processes with parameters 0.76 and 0.81, respectively, each using zero-mean unit variance white Gaussian driving noise. Then both signals were low-pass filtered to a max. frequency of $0.7 \cdot 2\pi/10$ using a Kaiser-windowed filter of 80 taps with window parameter $\beta = 3.44$ [1]. The signals were then delayed and mixed, prior to subsequent decimation by 10.

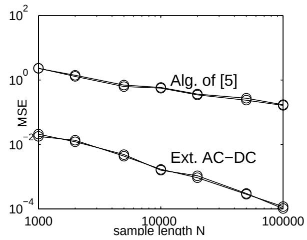

Fig. 1: delays estimation MSE

We used the mixing matrix $\boldsymbol{A} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, to enable comparison to the algorithm of [4], which can only accommodate this mixing matrix [2]. The results in terms of the mean squared error (mse) in estimating the delays vs. the sample length N are shown in figure 1. Both algorithms used the same data, with 100 trials for each sample length. Our algorithm was used with $K = 10$ matrices at 10 frequencies with $\Omega = 0.25$ spacing. The "pulsation parameter" for the algorithm of [4] was $\omega = 0.245$. The true nonzero delays were $\tau_{12} = 2.1, \tau_{21} = 5.5$.

5. REFERENCES

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[1] Providing for stop-band attenuation of about 40dB with a transition band of about $\pi/70$.

[2] Note also, that [4] implicitly uses the knowledge of this mixing matrix, whereas our algorithm attempted to estimate these mixing parameters as well.