A NEW GIBBS SAMPLING DOA ESTIMATOR BASED ON BAYESIAN METHOD

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ABSTRACT

A new Gibbs Sampling DOA estimator based on Bayesian method (GSDB) is proposed to estimate the directions of multiple sources. The estimator combines Gibbs sampler and the Bayesian high-resolution method. The formulation of the proposed Gibbs Sampling DOA estimator based on Bayesian method is derived. The new method not only possesses the performance of high-resolution direction finding in original Bayesian method but also provides reduced computational complexity to the original one from $O(L^K)$ to $O(K\times J\times N_s)$. Comparison with MUSIC shows that the new estimator has higher resolution and better performance in low SNRs.

1. INTRODUCTION

High-resolution DOA estimation is an important research area in array signal processing. It arises in many fields including sonar, radar, astronomy, radio communications and geophysics. DOA estimation has captured much attention in the past two decades, and many methods have been proposed for different applications. Eigen-decomposition based methods including MUSIC, Maximum Likelihood estimator (MLE), and MODE are some well-known procedures, and their performances have been thoroughly studied. In recent years, Bayesian high-resolution techniques [1], [2] and [3], which apply Bayes theorem in frequency and DOA estimation, become attractive for their superior performance. However the Bayesian high-resolution DOA estimators require multidimensional grid computation and search which are prohibitively expensive in the presence of large number of sources [2], [3]. In this paper, an algorithm combining the Bayesian method and the Gibbs sampler for DOA estimation is proposed. The proposed method provides notable improvement in the computational complexity over original Bayesian method we proposed with high-resolution performance.

2. BAYESIAN DOA ESTIMATOR

Consider a linear equi-spaced array of M sensors. The inter-element spacing b is equal to half of the carrier wavelength. Multiple far-field sources emit narrow-band signals with the direction parameters θ_k and frequencies f_k ($k=1,2,\dots,K$), which impinge on the sensors. These signals can be coherent or incoherent. The additive noise is assumed to be Gaussian and white with zero mean and variance σ^2. Let c denote the speed of the signal propagation in the medium, and $\tau_k = b\sin\theta_k / c$. Then the data collected from the m-th sensor at time t_n are

$$x_m(t_n) = \sum_{k=1}^{K} I_k(t_n)\exp[j\phi_k(t_n)]\exp[j2\pi f_k(t_n-(m-1)\tau_k)] + n_m(t_n)$$
$$= \sum_{k=1}^{K} A_k(t_n) f_{mk}(t_n) + n_m(t_n) \tag{1}$$

where $n=1,2,\dots,N$ with N being the number of snapshots, $A_k(t_n) = I_k(t_n)\exp[j\phi_k(t_n)]$, $I_k(t_n)$ is the unknown amplitude of the k-th signal at time t_n, $\phi_k(t_n)$ is the unknown phase of k-th signal at time t_n, $f_{mk}(t_n) = \exp[j2\pi f_k(t_n-(m-1)\tau_k)]$, and $n_m(t_n)$ is the noise at time t_n on the m-th sensor. Our main interest here is to estimate $\vec{\theta} = [\theta_1,\dots,\theta_K]^{\mathrm{T}}$. The unknown complex amplitudes $\vec{A} = \{A_k(t_n), \forall\, k, n\}$ and the noise variance σ^2 are considered as the nuisance parameters. From a Bayesian perspective, the main entity for estimation is the posterior distribution of θ which can be expressed as

$$p(\vec{\theta} \mid X) = \int p(X \mid \vec{\theta}, \vec{A}, \sigma^2)\, p(\vec{\theta}, \vec{A}, \sigma)\, d\vec{A}\, d\sigma^2 \tag{2}$$

To solve the integration analytically, an orthogonalization on the data snapshots is performed [2], [3]. In particular, first the snapshots are divided into N_b blocks with each block having n_b snapshots. Then the orthogonalization of the data in the s-th block is accomplished by

$$\sum_{k=1}^{K} A_k(t_n) f_{mk}(t_n) = \sum_{k=1}^{K} B_k H_{mk}(t_n) \quad (3)$$

where

$$H_{mk}(t_n) = \frac{1}{\sqrt{\lambda_k}} \sum_{l=1}^{K} e_{lk}^{*} f_{ml}(t_n), \qquad B_k = \sqrt{\lambda_k} \sum_{l=1}^{K} A_l e_{lk}, \qquad \text{and}$$

λ_k and $\vec{e}_k = [e_{1k}, e_{2k}, \cdots, e_{Kk}]^T$ are the eigenvalues and the eigenvectors of a $K \times K$ matrix F whose elements are defined as

$$F_{kl} = \sum_{n=1+(s-1)n_b}^{sn_b} \sum_{m=1}^{M} f_{mk}(t_n) f_{ml}^{*}(t_n). \quad (4)$$

Now, if the Jeffreys' priors are adopted, the desired posterior density function can be obtained as

$$p(\vec{\theta} \mid X) \propto \int \sigma^{2KN_b - 2MN - 1} \exp\left[-\frac{\left(\bar{d}^2 - \bar{h}^2\right)}{\sigma^2} \right] d\sigma$$

$$\propto \left[1 - \frac{\bar{h}^2}{\bar{d}^2} \right]^{(2KN_b - 2MN)/2} \quad (5)$$

where $$\bar{d}^2 = \sum_{s=1}^{N_b} \bar{d}_s^2 = \sum_{s=1}^{N_b} \sum_{n=1+(s-1)n_b}^{sn_b} \sum_{m=1}^{M} | x_m(t_n) |^2$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{M} | x_m(t_n) |^2 \quad (6)$$

and $$\bar{h}^2 = \sum_{s=1}^{N_b} \sum_{k=1}^{K} \left| \sum_{n=1+(s-1)n_b}^{sn_b} \sum_{m=1}^{M} x_m(t_n) H_{mk}^{*}(t_n) \right|^2 \quad (7)$$

Notice that (5) is highly nonlinear and high dimensional with respect to $\vec{\theta}$. Thus calculations of the popular Bayesian estimators could be very intensive, especially when K is large. For instance, to obtain the maximum *a posteriori* (MAP) estimator of $\vec{\theta}$, a K dimensional search is carried out to find the K maximum peaks in the posterior distribution. The angles corresponding to these K peaks are the MAP estimate of the directions of the sources. Suppose that L grids are used for each dimension. The complexity of the K dimensional computation is $O(L^K)$. Although the resolution ability of Bayesian method is rather high, the computational cost of the K dimensional computation and search could be prohibitively expensive for large K. To improve the real time computation of the Bayesian method, computational feasible solutions are demanded.

3. GIBBS SAMPLING

The Gibbs sampler is a Markov chain Monte Carlo (MCMC) sampling method for numerical evaluation of multidimensional integrations. Its popularity is gained from the facts that it is capable of carrying out many complex Bayesian computations. In the past decade, it has been intensively studied by statisticians and in recent years its applications in signal processing has been picked up.

The basic idea of the Gibbs sampler is to simulate a Markov chain in the state space of x so that the equilibrium of this chain is the target distribution $p(\vec{\theta} \mid X)$. So the Gibbs sampler algorithm is to first generate random samples from the joint posterior distribution $p(\vec{\theta} \mid X)$ by running Markov chains. Then the resulting samples are used by the Monte Carlo method to approximate the required high dimensional integrations. And the Gibbs sampler requires an initial transient period to converge to equilibrium. The initial period is known as the "burn-in" period, and the first n_0 samples in the period should always be discarded. Detection of convergence is usually done in some ad hoc way. For tutorials on the Gibbs sampler, see [4], [5].

4. BAYESIAN DOA ESTIMATOR BY GIBBS SAMPLING

We can notice that the high dimensional integrations in (2) impose great computational difficulty and the K dimensional search for the DOA estimation. To solve the real-time question, here we resort to the Gibbs sampler.

The key objective in a Gibbs sampling implementation is the generation of samples from the posterior distribution $p(\vec{\theta} \mid X)$. It is achieved through an iterative scheme. In a detail, given some initial values $\vec{\theta}^{(0)}$ of the K unknown directions, for $i = 1, 2, \cdots, N_s$, we proceed

1) Draw sample $\theta_1^{(i)}$ from $p\left(\theta_1 \mid \theta_2^{(i-1)}, \cdots, \theta_K^{(i-1)}, X\right)$

2) Draw sample $\theta_2^{(i)}$ from

$$p\left(\theta_2 \mid \theta_1^{(i)}, \theta_3^{(i-1)} \cdots, \theta_K^{(i-1)}, X\right)$$

K) Draw sample $\theta_K^{(i)}$ from $p\left(\theta_K \mid \theta_1^{(i)}, \cdots, \theta_{K-1}^{(i)}, X\right)$

Notice from (5) that $p(\theta_k \mid \theta_1^{(i)}, \cdots, \theta_{k-1}^{(i)}, \theta_{k+1}^{(i-1)}, \cdots, X)$ for all k=1, 2, … , K are not such distributions like the Gaussian or Gamma distributions. Therefore special care must be taken to achieve the required sampling objective. Next we proposed a procedure which applies the sampling-resampling [7] and kernel smoothing [7] techniques. In detail, the sampling of $\theta_k^{(i)}$ from $p(\theta_k \mid \theta_1^{(i)}, \cdots, \theta_{k-1}^{(i)}, \theta_{k+1}^{(i-1)}, \cdots, X)$ is carried out as follows:

1 Obtain J samples from the uniform distribution U(-90, 90)

and denoted them by $u(j),\ j=1,\cdots,J$.

2 For each $u(j)$, form a new vector $\bar{\alpha}_j^T=[\theta_1^{(i)},\theta_2^{(i)},\cdots,\theta_{k-1}^{(i)},u(j),\theta_{k+1}^{(i-1)},\cdots,\theta_K^{(i-1)}]$, and then calculate from the distribution (5) the weights

$$\bar{w}_j \propto p(\bar{\alpha}_j \mid X)$$

Next obtain the normalized weights by

$$w_j=\bar{w}_j \Big/ \sum_{j=1}^{J}\bar{w}_j .$$

3 Calculate the sample mean and variance according to

$$\bar{\mu}=\sum_{j=1}^{J} w_j u(j) \qquad \bar{\sigma}^2=\sum_{j=1}^{J} w_j (u(j)-\bar{\mu})^2$$

Then we approximate the conditional distribution $p(\theta_k \mid \theta_1^{(n)},\cdots,\theta_{k-1}^{(n)},\theta_{k+1}^{(n-1)},\cdots,X)$ by a mixture density as

$$g(\theta_k)=\sum_{j=1}^{J} w_j TN(\theta_k \mid \bar{\mu}_j, h^2\bar{\sigma}^2)$$

where $TN(\cdot \mid a,b)$ represents a truncated Gaussian with the mean a, the variance b, in particular, $h=(4/3)^{\frac{1}{5}} \big/ J^{\frac{-}{5}}$, $\beta=\sqrt{1-h^2}$, and $\mu_j=\beta\, u(j)+(1-\beta)\bar{\mu}$.

4 Now our original objective of sampling from $p(\theta_k \mid \theta_1^{(n)},\cdots,\theta_{k-1}^{(n)},\theta_{k+1}^{(n-1)},\cdots,X)$ becomes sampling from the mixture $g(\theta_k)$, $k=1,\cdots,K$ which is implemented as

(1) Sample an index j with probability w_j ,

(2) Sample $\theta_k^{(i)}$ from $TN(\theta_k \mid \bar{\mu}_j, h^2\bar{\sigma}^2)$.

To ensure convergence, the above procedure is usually carried out for (n_0+N) iterations, and samples from the last N iterations are used to calculate MMSE of the sources' directions. Finally, the MMSE of the directions of K sources can be obtained from the corresponding sample means as

$$E\{\theta_k \mid X\} \cong \frac{1}{N}\sum_{n=n_0+1}^{n_0+N}\theta_k^{(n)} \quad k=1,\cdots,K \tag{8}$$

The complexity of the proposed Gibbs Sampling DOA estimator based on Bayesian method (GSDB) is of $O(K\times J\times N_s)$. To compare with a K dimensional search, we observe that, first, due to the use of kernel smoothing techniques, J is smaller than L, the number of grid used in the K dimensional search. Secondly, as we will show next that the proposed algorithm converges very fast. Hence, N_s grow with K much slower than exponentially. As a result, for a large K, the computational demand of the GSDB is tremendously reduced with respect to that of the K dimensional search.

5. EXPERIMENTAL RESULTS

In this section, several experiments are conducted to show the performance of the GSDB. The performance comparisons with other popular methods such as MUSIC are also provided.

In first experiment, we consider a scenario of having three sources. The true DOAs of the sources are 72°, 76°and 80° . In Figure 1, we plotted three trajectories of the samples collected in the 80 iterations. It is clear that the samples of all signals converge very fast and fluctuate around the true values.

In the second experiment, the performance of the GSDB was compared with MUSIC and the results are shown in Table 1 and Table 2 (where BW means the bandwidth). These results are based on large amount of computer simulations and the statistical analysis indicates that the GSDB possesses the high resolution, and it is much more robust under low SNRs. From the two tables, we notice that, under high SNRs (Table1), the GSDB possesses higher estimation accuracy than other methods. And under low SNRs (Table2), the estimation accuracy of GSDB is still superior to that of MUSIC. When SNR=-5dB, MUSIC can not distinguish these three sources while the resolution probability of GSDB is 100%. And when the DOA interval between three sources becomes closer, the superiority of the new method is more obvious. Furthermore, the calculation is much less than the original Bayesian method. For example, for three sources the computation of original Bayesian method is about $O(10^9)$ while that of new method is only about $O(9*10^3)$. As we have shown above, the computation complexity of the original Bayesian method is $O(L^K)$.The accuracy could be improved as L increases while the computation complexity becomes also higher. And when K becomes bigger, the complexity will be increased exponentially. But the complexity of GSDB, $O(K\times J\times N_s)$, is increased only linearly as K increases with also keeping the original good performance. The comparison of computation complexity between the original Bayesian method and GSDB as K increases is shown in figure 2. It is clear that the GSDB is a very efficient. DOA high-resolution estimator for multiple source localization.

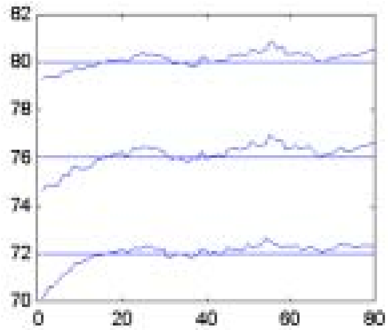

Figure1. Sample trajections of three signals from 72°, 76° and 80° are 80 iterations and the SNR is 15dB.

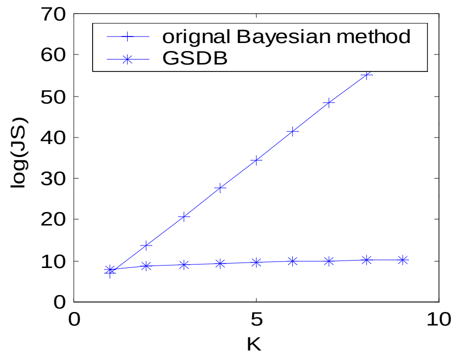

Figure2. The comparison of the computation complexity between original Bayesian method and GSDB, K is the number of sources, JS is the computation complexity. (J=1000, L=100, N_s =30)

Table 1 The comparison of angular resolution threshold between GSDB and MUSIC

Uniform Linear Array, M=12, BW=8.9104°, Snapshots N=100, SNR=15dB				
Method / Direction Of Sources	GSDB		MUSIC	
	Resolution Probability (%)	Root Mean Square Error (°)	Resolution Probability (%)	Root Mean Square Error (°)
$\theta_{1,2,3}$=72°,76°,78°	100	0.6689	66	0.8851

Table 2. The comparison among GSDB, MUSIC

Uniform Linear Array, M=12, BW=8.9104°, Snapshots N=500,								
Direction Of Sources / Method	SNR=0dB				SNR=-5dB			
	$\theta_{1,2,3}$=72°,76°,80°		$\theta_{1,2,3}$=72°,76°,78°		$\theta_{1,2,3}$=72°,76°,80°		$\theta_{1,2,3}$=72°,76°,78°	
	Resolution Probability (%)	Root Mean Square Error (°)	Resolution Probability (%)	Root Mean Square Error (°)	Resolution Probability (%)	Root Mean Square Error (°)	Resolution Probability (%)	Root Mean Square Error (°)
GSDB	100	0.8546	60	1.5693	60	1.8574	55	2.5912
MUSIC	10	1.0385	-	-	-	-	-	-

6. SUMMARY

In this paper a new method is presented which combines the Bayesian high-resolution DOA estimation with the Gibbs sampler. The formulation of the new method has been deduced and its promising performance has also been investigated. It has been shown that the new estimator possesses not only good performance but also improvement in reducing computational expenses from $O(L^K)$ to $O(K\times J\times N_s)$. The simulations have also demonstrated that it achieved higher resolution than MUSIC, especially in the low SNRs.

7. REFERENCES

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