ASYMPTOTICALLY OPTIMAL ESTIMATION OF DOA FOR NON-CIRCULAR SOURCES FROM SECOND ORDER MOMENTS

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ABSTRACT

This paper addresses asymptotically minimum variance (AMV) algorithm within the class of algorithms based on second-order statistics for estimating direction of arrival (DOA) parameters of possibly spatially correlated (even coherent) narrowband non-circular sources impinging on arbitrary array structures. To reduce the computational complexity due to the nonlinear minimization required by the matching approach, the covariance matching estimation techniques (COMET) is included in the algorithm. A numerical example illustrates the performance of the AMV algorithm.

1. INTRODUCTION

There is considerable literature about second-order statistics-based algorithms for estimating DOA of narrowband sources impinging on an array of sensors. The interest in these algorithms stems from a large number of applications including mobile communications systems [1]. In this application, after frequency down-shifting the sensor signals to baseband, the in-phase and quadrature components are paired to obtain complex signals. And complex non-circular signals, for example, binary phase shift keying (BPSK) modulated signals are often used. However, only a few contributions, such as [2],[3] have been devoted to non-circular signals.

The DOA second-order algorithms devoted to complex circular signals rely on the positive definite Hermitian covariance matrix $\mathrm{E}(\mathbf{y}_t\mathbf{y}_t^H)$, and naturally they can be used in the context of non-circular signals. Because, the second-order statistical characteristics are also contained in the complex symmetric covariance matrix $\mathrm{E}(\mathbf{y}_t\mathbf{y}_t^T)$ for non-circular signals, a potentially performance improvement ought to be obtained if these two covariance matrices are used. In the context of spatially uncorrelated amplitude modulated or BPSK modulated sources impinging on a linear uniform array, a significant performance improvement has been already observed by simulations in [2] and [3] thanks to a MUSIC-like algorithm and a root-MUSIC like algorithm respectively.

To improve the performance of these algorithms and to extend DOA estimation to spatially correlated or even coherent arbitrary non-circular sources and to arbitrary array structures, we propose to consider asymptotically (in the number of measurements) minimum variance algorithms in the class of algorithms based on the two covariance matrices. We extend to complex non-circular processes the result of Porat and Friedlander [4] devoted to the estimating of MA and ARMA parameters of real non-Gaussian processes from sample high-order statistics. After a general lower bound is derived for the covariance of the estimated DOAs, it is shown that a generalized covariance matching algorithm attains this bound. Furthermore, the ideas of COMET [5] are exploited to lower the dimensional optimization problem.

2. ASYMPTOTIC MINIMUM VARIANCE SECOND-ORDER ESTIMATOR

We consider a zero-mean strict-sense stationary M-variate complex, possibly non-circular process $\mathbf{y}_t$ whose structured covariance matrices $\mathbf{R}(\Theta) \stackrel{\mathrm{def}}{=} \mathrm{E}(\mathbf{y}_t\mathbf{y}_t^H)$ and $\mathbf{R}'(\Theta) \stackrel{\mathrm{def}}{=} \mathrm{E}(\mathbf{y}_t\mathbf{y}_t^T)$ are parameterized by the real parameter $\Theta \in \mathcal{R}^L$. This parameter is supposed identifiable from $(\mathbf{R}(\Theta), \mathbf{R}'(\Theta))$. These covariance matrices are classically estimated by $\mathbf{R}_T = \frac{1}{T}\sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t^H$ and $\mathbf{R}'_T = \frac{1}{T}\sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t^T$ respectively.

To extend the ideas of Porat and Friedlander [4] concerning asymptotically minimum variance second-order estimators, to complex non-circular processes, two conditions must be satisfied. First, the covariance $\mathbf{C}_{\substack{r\\r'}}(\Theta)$ of the asymptotic distribution of $(\mathbf{R}_T, \mathbf{R}'_T)$ must be regular. Second, the involved second-order algorithm considered as a mapping which associates to $(\mathbf{R}_T, \mathbf{R}'_T)$, the estimate Θ_T

$$(\mathbf{R}_T, \mathbf{R}'_T) \to \Theta_T = \mathrm{alg}(\mathbf{R}_T, \mathbf{R}'_T)$$

must be complex differentiable w.r.t. $(\mathbf{R}_T, \mathbf{R}'_T)$ at the point $(\mathbf{R}(\Theta), \mathbf{R}'(\Theta))$. While these two conditions are satisfied for a second-order based on $\mathbf{R}_T$ only, none of these two conditions are satisfied in our situation for the following reasons. First, because $\mathbf{R}'_T$ is symmetric, the rank of $\mathbf{C}_{\substack{r\\r'}}(\Theta)$

which is the rank of the set of the entries of $(\mathbf{R}_T, \mathbf{R}'_T)$ is not full. Consequently $\mathbf{C}_{\substack{r \\ r'}}(\Theta)$ is singular. Second, because $\mathbf{R}'_T$ is complex non Hermitian, an algorithm considered as a mapping, is not complex differentiable w.r.t. $\mathbf{R}'_T$ at point $\mathbf{R}'(\Theta)$.

To satisfy these two conditions, we must eliminate the common terms in $\mathbf{R}'_T$ and add complex conjuguate associated terms. Below, we consider the equivalent to $(\mathbf{R}_T, \mathbf{R}'_T)$ statistics $\mathbf{s}_T$ constituted by $\mathbf{r}_T \stackrel{\text{def}}{=} \text{Vec}(\mathbf{R}_T)$, $\tilde{\mathbf{r}}'_T \stackrel{\text{def}}{=} \text{v}(\mathbf{R}'_T)$ and $\tilde{\mathbf{r}}'^*_T \stackrel{\text{def}}{=} \text{v}(\mathbf{R}'^*_T)$, (where $\text{v}(.)$ denote the operator obtained from $\text{Vec}(.)$ by eliminating all supradiagonal elements of the matrix),

$$\mathbf{s}_T \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{r}_T \\ \tilde{\mathbf{r}}'_T \\ \tilde{\mathbf{r}}'^*_T \end{pmatrix}.$$

So $\mathbf{s}^* = \mathbf{P}\mathbf{s}$, where $\mathbf{P}$ is a permutation matrix. Consequently, any algorithm differentiable w.r.t. $(\Re(\mathbf{s}), \Im(\mathbf{s}))$ becomes differentiable w.r.t. $\mathbf{s}$ alone if $\delta\mathbf{s}$ is structured as $\delta\mathbf{s} = \begin{pmatrix} \delta\mathbf{r} \\ \delta\tilde{\mathbf{r}} \\ \delta\tilde{\mathbf{r}}'^* \end{pmatrix}$, in which case

$$\begin{aligned} \text{alg}[\mathbf{s}(\Theta) + \delta\mathbf{s}] &= \text{alg}[\mathbf{s}(\Theta)] + [\mathbf{D}_s, \mathbf{D}_s^*] \begin{bmatrix} \delta\mathbf{s} \\ \delta\mathbf{s}^* \end{bmatrix} + o(\delta\mathbf{s}) \\ &= \Theta + \mathbf{D}_s^{\text{alg}}\delta\mathbf{s} + o(\delta\mathbf{s}) \end{aligned}$$

with $\mathbf{D}_s^{\text{alg}} \stackrel{\text{def}}{=} \mathbf{D}_s + \mathbf{D}_s^*\mathbf{K}$, with $\mathbf{K}$ is the commutation matrix which transforms $\text{Vec}(.)$ into $\text{Vec}(.^T)$ for all $M \times M$ matrix. And because $\text{alg}[\mathbf{s}(\Theta)] = \Theta$ for all Θ:

$$\begin{aligned} \text{alg}[\mathbf{s}(\Theta + \delta\Theta)] &= \text{alg}[\mathbf{s}(\Theta) + \mathbf{S}\delta\Theta + o(\delta\Theta)] \\ &= \Theta + \mathbf{D}_s^{\text{alg}}\mathbf{S}\delta\Theta + o(\delta\Theta) = \Theta + \delta\Theta. \end{aligned}$$

Therefore $\mathbf{D}_s^{\text{alg}}$ is a left inverse of $\mathbf{S} \stackrel{\text{def}}{=} \frac{d\mathbf{s}(\Theta)}{d\Theta}$:

$$\mathbf{D}_s^{\text{alg}}\mathbf{S} = \mathbf{I}_L, \tag{1}$$

and this time, the rank of the set of the entries of $\mathbf{s}_T$ is full and so, the covariance $\mathbf{C}_s(\Theta)$ of the asymptotic distribution of $\mathbf{s}_T$ is a Hermitian positive definite matrix. Therefore, if $\mathbf{S}$ is a full column matrix, we prove [6] by application of theorem 2 of [4], extended to the complex case:

Theorem 1 *The asymptotic covariance of an estimator of Θ given by an arbitrary second-order algorithm is bounded below by $(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}$:*

$$\mathbf{C}_\Theta = \mathbf{D}_s^{\text{alg}}\mathbf{C}_s(\Theta)(\mathbf{D}_s^{\text{alg}})^H \geq (\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}. \tag{2}$$

Furthermore, we prove [6] that this lowest bound is asymptotically tight, i.e., there exists an algorithm alg(.) whose covariance of the asymptotic distribution of Θ_T satisfies (2) with equality. Therefore, theorem 3 of [4] extends to the complex non-circular case.

Theorem 2 *The following nonlinear least mean square algorithm is an AMV second-order algorithm.*

$$\Theta_T = \arg\min_\alpha [\mathbf{s}_T - \mathbf{s}(\alpha)]^H\mathbf{C}_s^{-1}(\alpha)[\mathbf{s}_T - \mathbf{s}(\alpha)]. \tag{3}$$

In practice, it is difficult to optimize the nonlinear function (3) where it involves the computation of $\mathbf{C}_s^{-1}(\alpha)$. Porat and Friedlander proved for the real case in [4], that the lowest bound (2) is also obtained if an arbitrary consistent estimate $\mathbf{C}_{s,T}$ of $\mathbf{C}_s(\alpha)$ is used in (3). This property extends to the complex non-circular case and to any Hermitian positive definite weighting matrix. And we prove [6]:

Theorem 3 *The covariance of the asymptotic distribution of Θ_T given by an arbitrary nonlinear least square algorithm*

$$\Theta_T = \arg\min_\alpha [\mathbf{s}_T - \mathbf{s}(\alpha)]^H\mathbf{W}(\alpha)[\mathbf{s}_T - \mathbf{s}(\alpha)],$$

is preserved if the Hermitian positive definite weighting matrix $\mathbf{W}(\alpha)$ is replaced by an arbitrary consistent estimate $\mathbf{W}_T$ that satisfies $\mathbf{W}_T = \mathbf{W}(\Theta) + O(\mathbf{s}_T - \mathbf{s}(\Theta))$.

So the minimization (3) can be preferably replaced by the following

$$\Theta_T = \arg\min_\alpha [\mathbf{s}_T - \mathbf{s}(\alpha)]^H\mathbf{C}_{s,T}^{-1}[\mathbf{s}_T - \mathbf{s}(\alpha)]. \tag{4}$$

3. APPLICATION TO ESTIMATION OF DOA

In the following, we will be concerned with the signal model

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{n}_t, \qquad t = 1, \ldots, T$$

where $(\mathbf{y}_t)_{t=1,\ldots,T}$ represents the independent identically distributed M-vectors of observed complex envelope at the sensor output. $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_K]$ is the steering matrix where each vector $\mathbf{a}_k$ is parameterized by the scalar parameter θ_k to avoid unnecessary notational complexity. But the results presented here apply to a general parameterization. $\mathbf{x}_t = (x_{t,1}, \ldots, x_{t,K})^T$ and $\mathbf{n}_t$ model signals transmitted by K sources and additive measurement noise respectively. $\mathbf{x}_t$ and $\mathbf{n}_t$ are multivariate independent, zero-mean, complex wide-sense stationary. $\mathbf{n}_t$ is assumed Gaussian complex circular, spatially uncorrelated with $\text{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma_n^2\mathbf{I}_M$, while $\mathbf{x}_t$ is complex circular or not, Gaussian or not and possibly spatially correlated or even coherent with $\mathbf{R}_x \stackrel{\text{def}}{=} \text{E}(\mathbf{x}_t\mathbf{x}_t^H)$ and $\mathbf{R}'_x \stackrel{\text{def}}{=} \text{E}(\mathbf{x}_t\mathbf{x}_t^T)$. Consequently this leads to the covariance matrices of $\mathbf{y}_t$:

$$\mathbf{R}(\Theta) = \mathbf{A}\mathbf{R}_x\mathbf{A}^H + \sigma_n^2\mathbf{I}_M \quad \text{and} \quad \mathbf{R}'(\Theta) = \mathbf{A}\mathbf{R}'_x\mathbf{A}^T.$$

$(\mathbf{R}(\Theta), \mathbf{R}'(\Theta))$ is generically parametrized by the $L = K + K^2 + K(K+1) + 1$ real parameters

$\Theta = (\Theta_1, \Theta_2)$ with $\Theta_1 \stackrel{\text{def}}{=} (\theta_1, \ldots, \theta_K)^T$ and $\Theta_2 \stackrel{\text{def}}{=} ((\Re([\mathbf{R}_x]_{i,j}), \Im([\mathbf{R}_x]_{i,j}), \Re([\mathbf{R}'_x]_{i,j}), \Im([\mathbf{R}'_x]_{i,j}))_{1 \leq j < i \leq K}, ([\mathbf{R}_x]_{i,i}, \Re([\mathbf{R}'_x]_{i,i}), \Im([\mathbf{R}'_x]_{i,i}))_{i=1,\ldots,K}, \sigma_n^2)^T$.

For performance analysis, some extra hypotheses are needed. The rank of $\mathbf{R}_x$ is denoted $\tilde{K}$. Clearly $\tilde{K} \leq K$, and strict inequality implies linear dependence among the signal waveforms emanating from, e.g., specular multipath or smart jamming in communication applications. We suppose that the signal waveforms are linearly issued from $\tilde{K}$ independent signals $(\tilde{x}_{t,k})_{k=1,\ldots,\tilde{K}}$, i.e., there exists a full colomn rank matrix $\mathbf{B}$ such that $\mathbf{x}_t = \mathbf{B}\tilde{\mathbf{x}}_t$. The fourth-order cumulants of these sources are denoted by $\kappa_{\tilde{x}_k} \stackrel{\text{def}}{=} \text{Cum}(\tilde{x}_{t,k}, \tilde{x}^*_{t,k}, \tilde{x}_{t,k}, \tilde{x}^*_{t,k})$, $\kappa'_{\tilde{x}_k} \stackrel{\text{def}}{=} \text{Cum}(\tilde{x}_{t,k}, \tilde{x}_{t,k}, \tilde{x}_{t,k}, \tilde{x}_{t,k})$ and $\kappa''_{\tilde{x}_k} \stackrel{\text{def}}{=} \text{Cum}(\tilde{x}_{t,k}, \tilde{x}^*_{t,k}, \tilde{x}^*_{t,k}, \tilde{x}^*_{t,k})$.

We note that $\mathbf{s}(\Theta)$ is linear with respect to Θ_2. Consequently there exists a known matrix $\Psi(\Theta_1)$ of the unknown DOA parameters Θ_1:

$$\mathbf{s}(\Theta) = \Psi(\Theta_1)\Theta_2. \tag{5}$$

Because, we suppose in this paper that Θ is identifiable from $(\mathbf{R}(\Theta), \mathbf{R}'(\Theta))$, Θ must be identifiable from $\mathbf{s}(\Theta)$, and necessarily $\Psi(\Theta_1)$ has column full rank. In these conditions, the minimization (4) with respect to Θ_2 is immediate if Θ_2 is not restricted to be real. With a geometric procedure, we obtain:

$$\widehat{\Theta}_2 = [\Psi^H(\Theta_1)\mathbf{W}\Psi(\Theta_1)]^{-1}\Psi^H(\Theta_1)\mathbf{W}\mathbf{s}_T \tag{6}$$

with $\mathbf{W} \stackrel{\text{def}}{=} \mathbf{C}_{s,T}^{-1}$. Because

$$\mathbf{s}_T = \frac{1}{T}\sum_{t=1}^{T}\mathbf{s}(t) \;\text{ with }\; \mathbf{s}(t) \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{y}_t^* \otimes \mathbf{y}_t \\ \mathbf{U}(\mathbf{y}_t \otimes \mathbf{y}_t) \\ \mathbf{U}(\mathbf{y}_t^* \otimes \mathbf{y}_t^*) \end{pmatrix},$$

where $\mathbf{U}$ is the selection matrix that satisfies $\text{v}(.) = \mathbf{U}\text{Vec}(.)$ for all $M \times M$ matrices, $\mathbf{s}_T$ is the mean of the T independent equidistributed random variables $\mathbf{s}(t)$. Consequently $\text{Cov}(\mathbf{s}_T) = \frac{1}{T}\text{Cov}(\mathbf{s}(t)) = \frac{1}{T}\text{E}\left[(\mathbf{s}(t) - \text{E}(\mathbf{s}(t)))(\mathbf{s}(t) - \text{E}(\mathbf{s}(t)))^H\right]$ and $\mathbf{C}_{s,T} = \frac{1}{T}\sum_{t=1}^{T}\left[\left(\mathbf{s}(t) - \frac{1}{T}\sum_{t=1}^{T}\mathbf{s}(t)\right)\left(\mathbf{s}(t) - \frac{1}{T}\sum_{t=1}^{T}\mathbf{s}(t)\right)^H\right]$ is a consistent estimate of $\mathbf{C}_s(\Theta)$ structured as $\mathbf{s}_T\mathbf{s}_T^H$. With arguments similar to that of COMET [5], we prove [6] that $\widehat{\Theta}_2$ is real-valued. Thus $\widehat{\Theta}_2$ given by (6) is the real value that minimizes (4). $\Theta_{1,T}$ is obtained by subtituting $\widehat{\Theta}_2$ in (4):

$$\Theta_{1,T} = \arg\max_{\alpha_1} V'(\alpha_1) \tag{7}$$

with $V'(\alpha_1) \stackrel{\text{def}}{=} \mathbf{s}_T^H\mathbf{W}\Psi(\alpha_1)[\Psi^H(\alpha_1)\mathbf{W}\Psi(\alpha_1)]^{-1}\Psi^H(\alpha_1)\mathbf{W}\mathbf{s}_T$. To evaluate the improvement provided by the use of the covariance matrix $\mathbf{R}'(\Theta)$, we first consider AMV second-order algorithms based on $\mathbf{R}_T$ only.

4. PERFORMANCE ANALYSIS

4.1. AMV estimator based on $\mathbf{R}_T$

We suppose here that Θ is identifiable from $\mathbf{R}(\Theta)$ only. In this case, the asymptotic minimum variance of the estimated parameters relies on the following standard central limit theorem applied to the independent equidistributed complex non-circular random variables $\mathbf{y}_t^* \otimes \mathbf{y}_t$. We prove in [6]:

Theorem 4 $\sqrt{T}\;(\text{Vec}(\mathbf{R}_T) - \text{Vec}(\mathbf{R}(\Theta)))$ *converges in distribution to the zero-mean complex non-circular Gaussian distribution of covariances* $\mathbf{C}_r$ *and* $\mathbf{C}'_r = \mathbf{C}_r\mathbf{K}$, *where*

$$\begin{aligned} \mathbf{C}_r &= (\mathbf{A}^* \otimes \mathbf{A})\mathbf{C}_{r_x}(\mathbf{A}^T \otimes \mathbf{A}^H) + \sigma_n^4\mathbf{I}_{M^2} \\ &+ \sigma_n^2\mathbf{I}_M \otimes \mathbf{A}\mathbf{R}_x\mathbf{A}^H + \mathbf{A}^*\mathbf{R}_x^*\mathbf{A}^T \otimes \sigma_n^2\mathbf{I}_M \end{aligned} \tag{8}$$

with $\mathbf{C}_{r_x} = \mathbf{R}_x^* \otimes \mathbf{R}_x + \mathbf{K}(\mathbf{R}'_x \otimes \mathbf{R}'^*_x) + \mathbf{Q}_x$ *and* $\mathbf{Q}_x = (\mathbf{B}^* \otimes \mathbf{B})\left(\sum_{k=1}^{\tilde{K}}\kappa_{\tilde{x}_k}(\mathbf{e}_{\tilde{K},k} \otimes \mathbf{e}_{\tilde{K},k})(\mathbf{e}_{\tilde{K},k}^T \otimes \mathbf{e}_{\tilde{K},k}^T)\right)(\mathbf{B}^T \otimes \mathbf{B}^H)$.

By application of theorem 1, the covariance of the asymptotic distribution of the minimum variance second-order DOA estimator (7) based on $\mathbf{R}_T$ only is given by the top left $K \times K$ "DOA corner" of $(\mathbf{S}^H\mathbf{C}_r^{-1}(\Theta)\mathbf{S})^{-1}$ where $\mathbf{C}_r(\Theta)$ is given by (8). If we note here that $\mathbf{S} \stackrel{\text{def}}{=} \frac{d\mathbf{r}}{d\Theta} = [\mathbf{S}_1, \Psi]$ with $\mathbf{S}_1 \stackrel{\text{def}}{=} \frac{\partial\mathbf{r}}{\partial\Theta_1}$ and Ψ given by $\mathbf{r} = \Psi(\Theta_1)\Theta_2$, the matrix inversion lemma gives

$$\begin{aligned} \mathbf{C}_{\Theta_1} &= (\mathbf{S}_1^H\mathbf{C}_r^{-1}\mathbf{S}_1 - \mathbf{S}_1^H\mathbf{C}_r^{-1}\Psi\left[\Psi^H\mathbf{C}_r^{-1}\Psi\right]^{-1} \\ &\qquad \Psi^H\mathbf{C}_r^{-1}\mathbf{S}_1)^{-1} \\ &= \left(\mathbf{S}_1^H\mathbf{C}_r^{-1/2}\mathbf{P}^{\perp}_{\mathbf{C}_r^{-1/2}\Psi}\mathbf{C}_r^{-1/2}\mathbf{S}_1\right)^{-1}, \end{aligned} \tag{9}$$

where $\mathbf{P}^{\perp}_{\mathbf{C}_r^{-1/2}\Psi}$ denotes the projector onto the orthocomplement of the colomns of $\mathbf{C}_r^{-1/2}\Psi$.

Because this AMV estimator does not suppose the sources spatially uncorrelated, its derivative $\mathbf{D}_r^{\text{AMV}}$ satisfies the constraint $\mathbf{D}_r^{\text{AMV}}(\mathbf{A}^* \otimes \mathbf{A}) = \mathbf{O}$ thanks to a lemma proved in [7]. Consequently the contribution of the first term of $\mathbf{C}_r$ (8) is canceled in the expression of the covariance $\mathbf{C}_\Theta = \mathbf{D}_r^{\text{AMV}}\mathbf{C}_r(\Theta)(\mathbf{D}_r^{\text{AMV}})^H$. Therefore, if we eliminate $\mathbf{K}(\mathbf{R}'_x \otimes \mathbf{R}'^*_x) + \mathbf{Q}_x$ from $\mathbf{C}_{r_x}$ only, $\mathbf{C}_r$ in (9) can be replaced by $\mathbf{R}^*(\Theta) \otimes \mathbf{R}(\Theta)$. And the expression (9) where $\mathbf{C}_r = \mathbf{R}^*(\Theta) \otimes \mathbf{R}(\Theta)$ extends to non Gaussian and/or complex non-circular sources, the expression of the asymptotic covariance given in [5] for Gaussian complex circular sources.

4.2. AMV estimator based on $(\mathbf{R}_T, \mathbf{R}'_T)$

The standard central limit theorem of the previous section extends similarly to the independent equidistributed complex non-circular random variables $\begin{bmatrix} \mathbf{y}_t^* \otimes \mathbf{y}_t \\ \mathbf{y}_t \otimes \mathbf{y}_t \end{bmatrix}$. We prove in [6]:

Theorem 5 $\sqrt{T}\begin{pmatrix}\mathrm{Vec}(\mathbf{R}_T)-\mathrm{Vec}(\mathbf{R}(\Theta))\\ \mathrm{Vec}(\mathbf{R}'_T)-\mathrm{Vec}(\mathbf{R}'(\Theta))\end{pmatrix}$ *converges in distribution to the zero-mean complex non-circular Gaussian distribution of covariances* $\mathbf{C}_{\substack{r\\r'}} = \begin{pmatrix}\mathbf{C}_r & \mathbf{C}_{r,r'}\\ \mathbf{C}^H_{r,r'} & \mathbf{C}_{r'}\end{pmatrix}$ *and* $\mathbf{C}'_{\substack{r\\r'}} = \begin{pmatrix}\mathbf{C}_r\mathbf{K} & \mathbf{K}\mathbf{C}^*_{r,r'}\\ \mathbf{C}^H_{r,r'}\mathbf{K} & \mathbf{C}'_{r'}\end{pmatrix}$ *where* $\mathbf{C}_r$ *is given by* (8) *and*

$$\begin{aligned}\mathbf{C}_{r'} &= (\mathbf{A}\otimes\mathbf{A})\mathbf{C}_{r'_x}(\mathbf{A}^H\otimes\mathbf{A}^H)+\sigma_n^4(\mathbf{I}_{M^2}+\mathbf{K})\\ &+ (\mathbf{I}_{M^2}+\mathbf{K})(\sigma_n^2\mathbf{I}_M\otimes\mathbf{A}\mathbf{R}_x\mathbf{A}^H\\ &+ \mathbf{A}\mathbf{R}_x\mathbf{A}^H\otimes\sigma_n^2\mathbf{I}_M) \qquad (10)\\ \mathbf{C}'_{r'} &= (\mathbf{A}\otimes\mathbf{A})\mathbf{C}'_{r'_x}(\mathbf{A}^T\otimes\mathbf{A}^T) \qquad (11)\\ \mathbf{C}_{r,r'} &= (\mathbf{A}^*\otimes\mathbf{A})\mathbf{C}_{r_x,r'_x}(\mathbf{A}^H\otimes\mathbf{A}^H) \qquad (12)\end{aligned}$$

with

$$\begin{aligned}\mathbf{C}_{r'_x} &= \mathbf{R}_x\otimes\mathbf{R}_x+\mathbf{K}(\mathbf{R}_x\otimes\mathbf{R}_x)+\mathbf{Q}_x\\ \mathbf{C}'_{r'_x} &= \mathbf{R}'_x\otimes\mathbf{R}'_x+\mathbf{K}(\mathbf{R}'_x\otimes\mathbf{R}'_x)+\mathbf{Q}'_x\\ \mathbf{C}_{r_x,r'_x} &= \mathbf{R}'^*_x\otimes\mathbf{R}_x+\mathbf{K}(\mathbf{R}_x\otimes\mathbf{R}'^*_x)+\mathbf{Q}''_x\end{aligned}$$

where $\mathbf{Q}_x$ *is given in theorem 4 and* $\mathbf{Q}'_x$ *and* $\mathbf{Q}''_x$ *are defined similarly.*

Then the asymptotic behavior of $\mathbf{s}_T$ and $(\mathbf{R}_T,\mathbf{R}'_T)$ are directly related by the standard continuity theorem. Therefore:

$$\sqrt{T}\ (\mathbf{s}_T-\mathbf{s}(\Theta)) \xrightarrow{\mathcal{L}} \mathcal{N}_c\left(\mathbf{0};\mathbf{C}_s(\Theta),\mathbf{C}'_s(\Theta)\right)$$

with

$$\mathbf{C}_s(\Theta)=\begin{pmatrix}\mathbf{C}_r & \mathbf{C}_{r,r'}\mathbf{U}^T & \mathbf{K}\mathbf{C}^*_{r,r'}\mathbf{U}^T\\ \mathbf{U}\mathbf{C}^H_{r,r'} & \mathbf{U}\mathbf{C}_{r'}\mathbf{U}^T & \mathbf{U}\mathbf{C}'_{r'}\mathbf{U}^T\\ \mathbf{U}\mathbf{C}^T_{r,r'}\mathbf{K} & \mathbf{U}\mathbf{C}'^H_{r'}\mathbf{U}^T & \mathbf{U}\mathbf{C}^*_{r'}\mathbf{U}^T\end{pmatrix} \qquad (13)$$

and $\mathbf{C}'_s(\Theta)=\mathbf{C}_s(\Theta)\mathbf{P}$. The covariance of the asymptotic distribution of the minimum variance second-order DOA estimator (7) based on $(\mathbf{R}_T,\mathbf{R}'_T)$ is similarly given by (9) where here $\mathbf{S}_1 \stackrel{\text{def}}{=} \frac{\partial\mathbf{s}}{\partial\Theta_1}$, Ψ given by $\mathbf{s}=\Psi(\Theta_1)\Theta_2$ and $\mathbf{C}_r(\Theta)$ is replaced by $\mathbf{C}_s(\Theta)$ given in (13):

5. SIMULATIONS

In this section, an example is given to illustrate the expected benefit due to the non-circular property. This will give an indication of the information contributed by the second covariance matrix. Two sources emit equipowered and spatially uncorrelated unfiltered BPSK modulated signals, and the number of data samples is $T=200$. We consider a uniform linear array of $M=6$ sensors separated by a half-wavelength for which $\mathbf{a}_k=(1,e^{i\theta_k},\ldots,e^{i(M-1)\theta_k})^T$.

Fig.1 exhibits the theoretical normalized asymptotic variance $[\mathbf{C}_{\Theta_1}]_{1,1}$ given by the AMV estimator based on $\mathbf{R}_T$ only and the AMV estimator based on $(\mathbf{R}_T,\mathbf{R}'_T)$, versus the DOA separation for a SNR of 10dB. The AMV estimator based on $(\mathbf{R}_T,\mathbf{R}'_T)$ clearly outperforms the AMV estimator based on $\mathbf{R}_T$ only, and the difference is particularly prominent when the sources are very close.

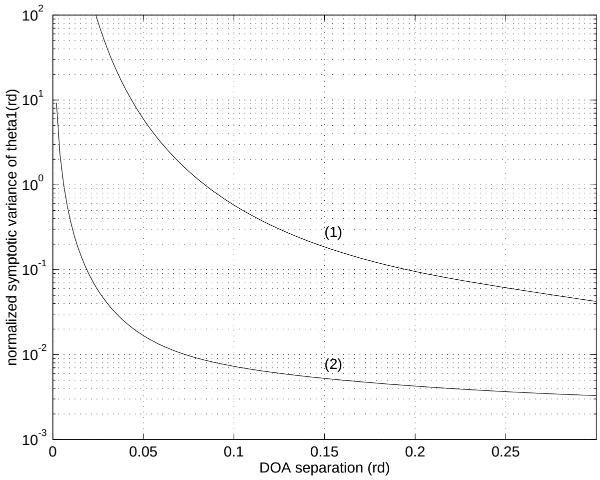

Fig.1 Theoretical normalized asymptotic variance of $\theta_{1,T}$ ($[\mathbf{C}_{\Theta_1}]_{1,1}$) given by the AMV estimator based on ($\mathbf{R}_T$ only (1) and the AMV estimator based on $(\mathbf{R}_T,\mathbf{R}'_T)$ (2), versus the DOA separation.

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