AN UNBIASED ESTIMATOR FOR BEARINGS-ONLY TRACKING AND DOPPLER-BEARING TRACKING

K. C. Ho

University of Missouri-Columbia
Dept of Electrical & Computer Engineering
Columbia, MO 65211, USA

Y. T. Chan

The Chinese University of Hong Kong
Department of Electronic Engineering
Shatin, NT, Hong Kong

ABSTRACT

The objective of both Bearings-only tracking (BOT) and Doppler-bearing tracking (DBT) is to obtain the target trajectory based on bearings, and Doppler and bearing measurements respectively, from an observer to the target. The BOT and DBT problems are nontrivial because the measurement equations are nonlinear. The pseudo linear formulation allows a linear estimator to solve for the solution, but the solution obtained is biased. This paper proposes an estimator based on the pseudo linear equations that will produce an unbiased solution. The proposed method applies least-squares minimization on the pseudo linear equations with appropriate constraints on the unknown parameters. Simulations are included to illustrate the performance of the proposed estimator. The proposed estimator achieves the Cramer-Rao Lower Bound (CRLB) for Gaussian noise around small error region.

1. INTRODUCTION

The tracking of a target based on the bearing angles from an observer to the target is of considerable interests over the past several decades [1]-[4]. The bearings-only tracking (BOT) requires the target to be moving at a constant speed, while the observer must maneuver in order to ensure a determinant solution. When the source emits harmonic components, the harmonic signals will experience Doppler shifts at the observer so that the Doppler shift measurements can be explored to improve the estimation of the target velocity. The use of both Doppler shifts and bearing angles to track a moving target is termed as Doppler-bearing Tracking (DBT).

The measurement equations in both BOT and DBT are nonlinear, which makes the target tracking a nontrivial task. Lindgren and Gong [1] proposed a pseudo linear (PL) estimator for the BOT problem, where the nonlinearities are all lumped into the noise term so that the tracking problem can be solved by Kalman filtering technique. One undesirable effect in the PL formulation is that the noisy bearing angles appear in the measurement matrix so that it is correlated with the noise in the measurements. The consequence is the creation of bias in the solution [2]. The bias can be very significant, and it does not decrease as the number of measurement increases.

Several techniques have been proposed to overcome the bias in the BOT and DBT tracking problem [3]-[4]. Among these methods, instrumental variable (IV) is shown to be effective in removing the solution bias [4]. The IV iterative method is iterative and requires a good initial solution guess. If the initial solution guess is far from the true solution, the IV method will fail to converge. In fact, there is no guarantee that the IV method will converge to the correct solution even for good initial guesses, when the geometry between the target and observer becomes unfavorable as the target proceeds.

This paper gives an unbiased method based on the PL formulation to solve the BOT and DBT problem. The proposed technique is a constrained least-squares (LS) minimization of the PL equations. It does not require any initial solution guesses and therefore does not have the convergence problem as experienced in the IV method. The formulation leads to a generalized eigenvalue problem. Simulation shows that the proposed solution achieves the CRLB for Gaussian noise over small error region. In the following, Section 2 provides the unbiased solution to BOT, and Section 3 gives the solution method to the DBT. Section 4 presents simulations and Section 5 is the conclusion.

2. BEARINGS-ONLY TRACKING

Let $(\, x_T(0)\,,\, y_T(0)\,)$ be the initial position of the target at time 0. Assuming the target is moving at a constant speed $(\, \dot{x}_T\,,\, \dot{y}_T\,)$, then at time iT, the target's position is

$$\begin{aligned} x_T(i) &= x_T(0) + iT\,\dot{x}_T \\ y_T(i) &= y_T(0) + iT\,\dot{y}_T \end{aligned} \qquad (1)$$

where T is the observation period and i is the time index. Let $(\, x_o(i)\,,\, y_o(i)\,)$ be the coordinate of the observer at time index i. To determine the target trajectory, the observer measures the bearing

$$\beta_i = \bar{\beta}_i + e_{\beta,i} \qquad (2)$$

where $\bar{\beta}_i$ is the true bearing and $e_{\beta,i}$ is the measurement noise. Given the observer positions $(\, x_o(i)\,,\, y_o(i)\,)$ and the bearing measurements β_i, $i = 0,\, 1,\, \cdots\,,\, k$, we wish to obtain the target position at instant k, $(\, x_T(k)\,,\, y_T(k)\,)$. Since the tartget is moving at a constant speed, the problem becomes the estimation of the target's initial position $(\, x_T(0)\,,\, y_T(0)\,)$ and speed $(\, \dot{x}_T\,,\, \dot{y}_T\,)$.

As shown in Figure 1, the true bearing $\bar{\beta}_i$ is related to the target's initial position and velocity by

$$\frac{sin(\bar{\beta}_i)}{cos(\bar{\beta}_i)} = \frac{x_T(i) - x_o(i)}{y_T(i) - y_o(i)}\,. \qquad (3)$$

Cross multiplying gives

$$cos(\bar{\beta}_i)\,[x_T(i) - x_o(i)] - sin(\bar{\beta}_i)\,[y_T(i) - y_o(i)] = 0\,. \qquad (4)$$

Given the bearing measurements up to time index k, upon using (1), the target's inital position and velocity can be found by minimizing

$$J_0 = \sum_{i=0}^{k} \big(cos(\bar{\beta}_i)\,[x_T(0) + iT\dot{x}_T - x_o(i)] - sin(\bar{\beta}_i)\,[y_T(0) + iT\dot{y}_T - y_o(i)] \big)^2 = 0\,, \quad (5)$$

which will give an unbiased solution.

The true bearing angles are not known and only the noisy bearing meaurements are available. Replacing $\bar{\beta}_i$ by β_i in (4) results in the equation error

$$\epsilon_{\beta,i} = cos(\beta_i)[x_T(i) - x_o(i)] - sin(\beta_i)[y_T(i) - y_o(i)]. \quad (6)$$

Let $\boldsymbol{\mu} = [\,x_T(0)\,,\, \dot{x}_T\,,\, y_T(0)\,,\, \dot{y}_T\,]^T$ be the unknown parameter vector, $\mathbf{g} = [\,cos(\beta_0)x_o(0) - sin(\beta_0)y_o(0)\,,\, \cdots\,,\, cos(\beta_k)x_o(k) - sin(\beta_k)y_o(k)]^T$ be the measurement vector, and

$$\mathbf{A} = \begin{bmatrix} cos(\beta_0) & 0 & -sin(\beta_0) & 0 \\ cos(\beta_1) & Tcos(\beta_1) & -sin(\beta_1) & -Tsin(\beta_1) \\ \vdots & \vdots & \vdots & \vdots \\ cos(\beta_k) & kTcos(\beta_k) & -sin(\beta_k) & -kTsin(\beta_k) \end{bmatrix}$$

be the measurement matrix. Upon using (1) and collecting the equation errors in (6) as a vector $\boldsymbol{\epsilon} = [\,\epsilon_{\beta,0}\,,\, \epsilon_{\beta,1}\,,\, \cdots\,,\, \epsilon_{\beta,k}\,]^T$, we have

$$\boldsymbol{\epsilon} = \mathbf{A}\,\boldsymbol{\mu} - \mathbf{g}\,. \quad (7)$$

Minimizing $\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}$ with respect to $\boldsymbol{\mu}$ yields the least-squares solution

$$\boldsymbol{\mu} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1} \mathbf{A}^T\,\mathbf{g}. \quad (8)$$

This LS solution is biased. To see the reason, putting $\mathbf{g} = \mathbf{A}\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\epsilon}}$ into (8) and taking expectation yields

$$E[\boldsymbol{\mu}] = \bar{\boldsymbol{\mu}} - E\left[\left(\mathbf{A}^T\mathbf{A}\right)^{-1} \mathbf{A}^T\,\bar{\boldsymbol{\epsilon}}\right] \quad (9)$$

where $\bar{\boldsymbol{\mu}}$ is the true solution and $\bar{\boldsymbol{\epsilon}}$ is the equation error vector evaluated at $\bar{\boldsymbol{\mu}}$. Note that the noisy bearing measurements appear in both $\mathbf{A}$ and $\bar{\boldsymbol{\epsilon}}$ and they are correlated. As a result, the second term on the right of (9) is not zero and the solution (8) is biased. The bias can be very significant. It does not decrease as the number of bearing measurements increases and it increases with the measurement noise power.

We shall propose a method that will give an unbiased solution. The solution derivation begins by putting (2) in (6). Applying the trigonometric identities, we have

$$\begin{aligned} \epsilon_{\beta,i} &= cos(e_{\beta,i})[cos(\bar{\beta}_i)(x_T(i) - x_o(i)) \\ &\quad - sin(\bar{\beta}_i)(y_T(i) - y_o(i))] \\ &\quad - sin(e_{\beta,i})[sin(\bar{\beta}_i(x_T(i) - x_o(i)) \\ &\quad - cos(\bar{\beta}_i)(y_T(i) - y_o(i))] \\ &\approx [\,cos(\bar{\beta}_i)(x_T(i) - x_o(i)) \\ &\quad - sin(\bar{\beta}_i)(y_T(i) - y_o(i))\,] \\ &\quad - e_{\beta,i}[\,sin(\bar{\beta}_i(x_T(i) - x_o(i)) \\ &\quad - cos(\bar{\beta}_i)(y_T(i) - y_o(i))\,], \end{aligned} \quad (10)$$

where small bearing errors $e_{\beta,i}$ is assumed. Squaring (10), taking expectation and summing over i up to current instant k yield

$$\begin{aligned} &\sum_{i=0}^{k} E[\epsilon_{\beta,i}^2] \\ &= \sum_{i=0}^{k} [\,cos(\bar{\beta}_i)(x_T(i) - x_o(i)) - sin(\bar{\beta}_i)(y_T(i) - y_o(i))\,]^2 \\ &\quad + \sigma_{e,\beta}^2 \sum_{i=0}^{k} [\,sin(\bar{\beta}_i)(x_T(i) - x_o(i)) \\ &\qquad + cos(\bar{\beta}_i)(y_T(i) - y_o(i))\,]^2 \\ &= J_0 + \sigma_{e,\beta}^2\, J_1, \end{aligned} \quad (11)$$

where the first term J_0 is the same as that in (5) and $J_1 = \sum_{i=0}^{k} [sin(\bar{\beta}_i)\,(x_T(i) - x_o(i)) + cos(\bar{\beta}_i)(y_T(i) - y_o(i))]^2$. In order to find an unbiased estimator by minimizing $\sum_{i=0}^{k} E[\epsilon_i^2]$, J_1 has to be fixed at a special value.

J_1 can be kept constant by introducing one more degree of freedom in the unknown parameter vector. Let $\boldsymbol{\theta}$ be the augmented parameter vector that is different from $[\boldsymbol{\mu}\,,\, 1]^T$ by a scaling factor, i.e. $\boldsymbol{\theta} = h[\boldsymbol{\mu}\,,\, 1]^T$, where h is a constant. For simplicity, define

$$\begin{aligned} \mathbf{u}_i = &[\,sin(\bar{\beta}_i)\,,\, iTsin(\bar{\beta}_i)\,,\, cos(\bar{\beta}_i)\,,\, iTcos(\bar{\beta}_i)\,, \\ &-x_o(i)sin(\bar{\beta}_i) - y_o(i)cos(\bar{\beta}_i)\,]^T\,, \end{aligned} \quad (12)$$

then

$$J_1 = \sum_{i=0}^{k} \boldsymbol{\theta}^T\mathbf{u}_i{\mathbf{u}_i}^T\boldsymbol{\theta} = \boldsymbol{\theta}^T\,\mathbf{W}\boldsymbol{\theta} \quad (13)$$

where $\mathbf{W} = \sum_{i=0}^{k} \mathbf{u}_i{\mathbf{u}_i}^T$. Note that $\mathbf{W}$ has the true bearing angles that are not known. The true bearing angles can be replaced by the noisy angle measurements. The resulting error is negligible especially when the noise is small.

Let $\mathbf{A}_u = [\,\mathbf{A}\,,\, -\mathbf{g}\,]^T$, then $\boldsymbol{\epsilon} = \mathbf{A}_u\boldsymbol{\theta}/h$. In vector form, the solution $\boldsymbol{\theta}$ is obtained by minmiizing

$$\boldsymbol{\theta}^T\mathbf{A}_u^T\mathbf{A}_u\boldsymbol{\theta} \quad \text{subject to} \quad \boldsymbol{\theta}^T\mathbf{W}\boldsymbol{\theta} = 1. \quad (14)$$

The constant 1 in the constraint is arbitrary. Using a different value other than 1 simply changes the constant h in $\boldsymbol{\theta}$.

The constrained minimization problem can be solved by Lagrange multiplier method by forming the auxilary cost function

$$\xi = \boldsymbol{\theta}^T\mathbf{A}_u^T\mathbf{A}_u\boldsymbol{\theta} + \lambda\,(1 - \boldsymbol{\theta}^T\mathbf{W}\boldsymbol{\theta}) \quad (15)$$

where λ is the Lagrange multiplier. Taking partial derivative of ξ with respect to $\boldsymbol{\theta}$ gives

$$\mathbf{A}_u^T\mathbf{A}_u\boldsymbol{\theta} = \lambda\mathbf{W}\boldsymbol{\theta} \quad (16)$$

which indicates that $\boldsymbol{\theta}$ is the generalized eigenvector. Premultiplying (16) by $\boldsymbol{\theta}$ forms $\lambda = \boldsymbol{\theta}^T\mathbf{A}_u^T\mathbf{A}_u\boldsymbol{\theta}$ which is the quantity to be minimized. Hence $\boldsymbol{\theta}$ is the generalized eigenvector of the pair $\left(\mathbf{A}_u^T\mathbf{A}_u\,,\, \mathbf{W}\right)$ that gives the minimum generalized eigenvalue. Once $\boldsymbol{\theta}$ is found, then $h = \theta(5)$ and the unknown parameter vector is

$$\boldsymbol{\mu} = \frac{[\theta(1)\,,\, \theta(2)\,,\, \theta(3)\,,\, \theta(4)\,]^T}{\theta(5)}\,. \quad (17)$$

3. DOPPLER-BEARING TRACKING

When the target and observer have relative motions where the target's radial velocity with respect to the observer is v, then a tonal f_s from the target will experience a Doppler shift at the observer so that the measured frequency at time index i relative to the source frequency is

$$\frac{\bar{f}_i}{f_s} = \left(1 + \frac{v}{c}\right) \tag{18}$$

where c is the propagation speed of the tone signal. Expressing the radial velocity v in terms of the target and observer velocity [4], (18) becomes

$$\frac{\bar{f}_i}{f_s} = \left(1 - \frac{\dot{x}_T - \dot{x}_o(i)}{c} sin(\bar{\beta}_i) - \frac{\dot{y}_T - \dot{y}_o(i)}{c} cos(\bar{\beta}_i)\right). \tag{19}$$

Since the true Doppler shifts and bearing angles are not available, replacing them by the noisy measurements f_i and β_i gives the equation error

$$\begin{aligned} e_{f,i} &= \frac{f_i}{f_s} - \left(1 - \frac{\dot{x}_T - \dot{x}_o(i)}{c} sin(\beta_i) \right. \\ & \left. - \frac{\dot{y}_T - \dot{y}_o(i)}{c} cos(\beta_i)\right). \end{aligned} \tag{20}$$

Combining the Doppler and bearing measurement equations (20) and (6) yields

$$\boldsymbol{\epsilon}' = \mathbf{A}'\boldsymbol{\mu}' - \mathbf{g}' \tag{21}$$

$$\boldsymbol{\epsilon}' = [\, e_{f,0}\, ,\, e_{\beta,0}\, ,\, \cdots\, ,\, e_{f,k}\, ,\, e_{\beta,k}\,]^T$$

$$\boldsymbol{\mu}' = [1/f_s\, ,\, x_T(0)\, ,\, \dot{x}_T\, ,\, y_T(0)\, ,\, \dot{y}_T\,]^T$$

$$\mathbf{A}' = \begin{bmatrix} f_0 & 0 & \frac{sin(\beta_0)}{c} & 0 & \frac{cos(\beta_0)}{c} \\ 0 & cos(\beta_0) & 0 & -sin(\beta_0) & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ f_k & 0 & \frac{sin(\beta_k)}{c} & 0 & \frac{cos(\beta_k)}{c} \\ 0 & cos(\beta_k) & kTcos(\beta_k) & -sin(\beta_k) & -kTsin(\beta_k) \end{bmatrix}$$

$$\mathbf{g}' = \begin{bmatrix} 1 + \dot{x}_o(0)sin(\beta_0)/c + \dot{y}_o(0)cos(\beta_0)/c \\ x_o(0)cos(\beta_0) - y_o(0)sin(\beta_0) \\ 1 + \dot{x}_o(1)sin(\beta_1)/c + \dot{y}_o(1)cos(\beta_1)/c \\ x_o(1)cos(\beta_1) - y_o(1)sin(\beta_1) \\ \vdots \\ 1 + \dot{x}_o(k)sin(\beta_k)/c + \dot{y}_o(k)cos(\beta_k)/c \\ x_o(k)cos(\beta_k) - y_o(k)sin(\beta_k) \end{bmatrix}.$$

As in the BOT case, direct minimization of $\boldsymbol{\epsilon}'^T\boldsymbol{\epsilon}'$ will give a biased solution.

Following the same argument as in BOT, it can be shown that by putting (21) in the form $\mathbf{e}' = \mathbf{A}'_u\,\boldsymbol{\theta}'$, where $\mathbf{A}'_u = [\mathbf{A}' \; -\mathbf{g}']$ and $\boldsymbol{\theta}' = h'[\boldsymbol{\mu}'\, ,\, 1]^T$, then minimizing

$$\boldsymbol{\theta}'^T \mathbf{A}'^T_u \mathbf{A}'_u \boldsymbol{\theta}' \quad \text{subject to} \quad \boldsymbol{\theta}'^T \mathbf{W}' \boldsymbol{\theta}' = 1 \tag{22}$$

will give an unbiased solution. The constraint matrix $\mathbf{W}'$ in this case is given by

$$\mathbf{W}' = \begin{bmatrix} (k+1)\sigma_f^2 & 0 \\ 0 & \sigma_\beta^2 \sum_{i=0}^k \left(\mathbf{u}_i\mathbf{u}_i^T + \mathbf{v}_i\mathbf{v}_i^T\right) \end{bmatrix} \tag{23}$$

where $\mathbf{u}_i$ is defined in (12) and the vector $\mathbf{v}_i$ is defined as $\mathbf{v}_i = [\, 0\, ,\, cos(\bar{\beta}_i)/c\, ,\, 0\, ,\, -sin(\bar{\beta}_i)/c\, ,\, -\dot{x}_o(i)cos(\bar{\beta}_i)/c - \dot{y}_o(i)\, sin(\bar{\beta}_i)\, /c\,]^T$. Since the true bearing angles are not available, they will be replaced by the noisy bearing angles in $\mathbf{W}'$.

4. SIMULATION

The simulations for both BOT and DBT use the scenarios described in [4]. For BOT, the observer starts at (0,0) and is moving along a zig-zag path at a constant speed of 12.7 m/s as shown in Figure 2. It maneuvers at time instances k=50, 150, 250 and 350 with 90 degree turns. The target is initially at (12.7km, 12.7km) and is moving at a constant speed of 9 m/s at an angle of 45^o with respect to the x-axis. The observation period is $T = 2s$. The noise in the bearing measurements are Gaussian white with power equal to $\sigma_\beta^2 = (1^o)^2$.

Figures 3 and 4 give the root mean square errors (rmse) and the bias in the x-coordinate position and velocity of the target trajectory when the proposed method is used. The results show that the proposed method is unbiased, and the rmse converge close to the CRLB as the number of measurement increases. The results from the IV method are also included for comparison. The instrument in the IV method was initialized to the true solution at time k=100 to generate its most favorable results. While the proposed method produces good target trajectory estimate, the IV method diverges after k=200 and fails to converge to a solution.

For the DBT simulation, the observer is stationary at the origin. The target is initially at (20km, -18km) and proceeds due North at 9 m/s radiating a single 300Hz tone. The observation period in this simulation is $T = 5s$. The frequency measurements are corrupted with zero mean Gaussian noise of power $\sigma_f^2 = (0.2Hz)^2$, and the Gaussian noise in the bearing measurements has a power of $\sigma_\beta^2 = (1^o)^2$. Figures 5 and 6 give the results in the x-coordinate position and velocity estimates of the target using the proposed method. Also given are the results from the IV method with initialization to the true solution at $k = 100$. It is clear that the proposed method is able to track the target well and produce position estimates close to the CRLB. This is in contrast with the IV method where divergence occured frequently.

5. CONCLUSION

This paper proposes an unbiased estimator to the constant velocity target trajectory for the BOT and DBT problems. It applies a constrained least-squares minimization on the pseudo linear measurement equations to obtain an unbiased solution. The problem is shown to be equal to solving the generalized eigenvector of the augmented measurement matrix that has the smallest generalized eigenvalue. The new method does not require initial solution guesses as in the instrumental variable method and therefore avoids divergence problems. Simulation confirms that the proposed estimator produces unbiased solution, and asymptotically achieves accuracy close to the CRLB for Gaussian noise.

6. REFERENCES

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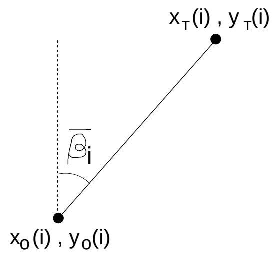

Fig. 1. Bearing angle, observer and target postions.

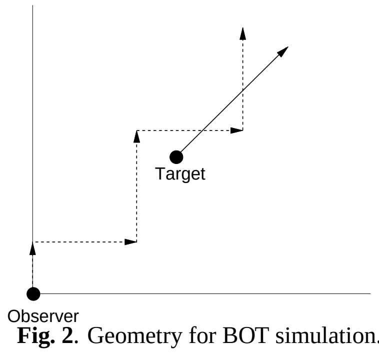

Fig. 2. Geometry for BOT simulation.

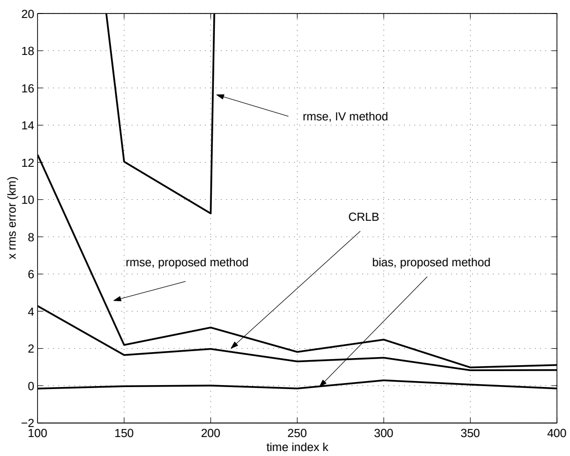

Fig. 3. Performance of the proposed method in BOT, x-position, $\sigma_\beta = 1^o$.

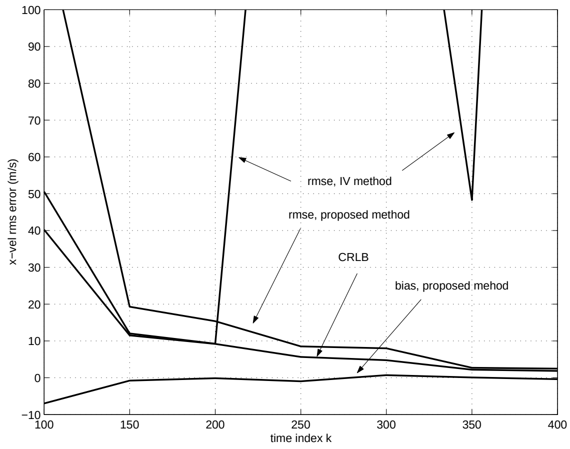

Fig. 4. Performance of the proposed method in BOT, x-velocity, $\sigma_\beta = 1^o$.

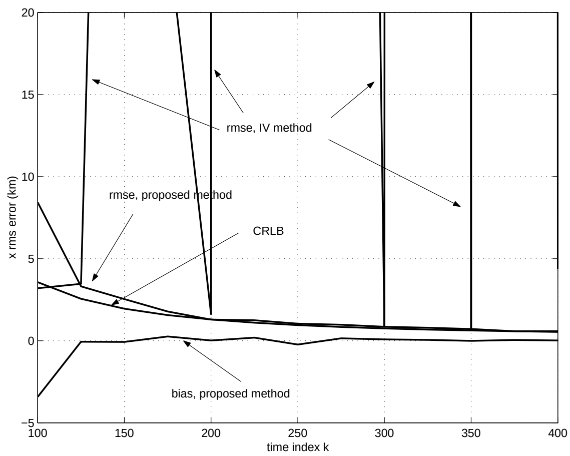

Fig. 5. Performance of the proposed method in DBT, x-position, $\sigma_f = 0.2Hz$, $\sigma_\beta = 1^o$.

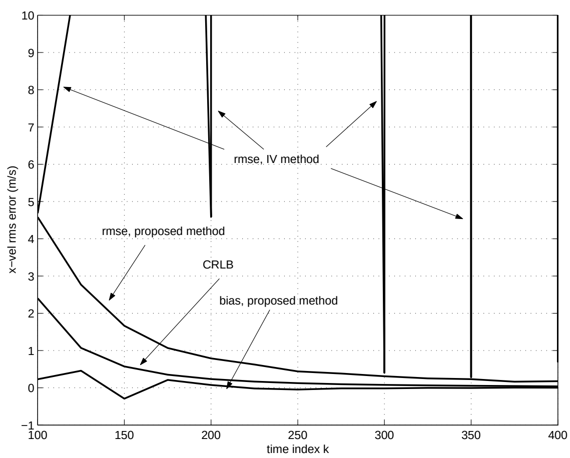

Fig. 6. Performance of the proposed method in DBT, x-velocity, $\sigma_f = 0.2Hz$, $\sigma_\beta = 1^o$.