THROUGHPUT MAXIMIZATION FOR THE MULTIUSER MIMO BROADCAST CHANNEL

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ABSTRACT

We consider the problem of maximizing the throughput (sum rate) of the Gaussian MIMO broadcast channel under a sum power constraint. Assuming that decision feedback precoding is used at the transmitter, this is a concave optimization problem. In [1], a computationally efficient algorithm was proposed, which successively performs iterative waterfilling and power control. This strategy is based on uplink/downlink duality. In this paper we expand these results, by providing necessary and sufficient conditions for when this dedicated strategy achieves the optimal sum capacity. For the low SNR regime, it is shown that the capacity achieving strategy is single user transmission over the channel with the largest maximal eigenvalue. Furthermore, we illustrate the properties of the sum capacity without precoding by numerical simulations.

1. INTRODUCTION

Consider a Gaussian broadcast channel (BC) with K non-cooperating receivers, each equipped with n_r antennas. The transmitter has n_t antennas. Perfect channel side information is available at the transmitting base station. The total transmission power (sum power) is upper bounded. Such a channel belongs to the class of non-degraded broadcast channels for which the general capacity region is not yet known.

A partial solution was found by Caire and Shamai [2] for the special case of two users, two transmit antennas and one receive antenna per user. The result shows that throughput-wise optimal transmission is possible by using a combination of linear pre-filtering and coding for non-causally known interference at the transmitter ('Dirty Paper' Precoding) [3,4]. In [5], the throughput-wise optimality of this approach was shown for arbitrary numbers of antennas. Dirty Paper Precoding decomposes the channel into a series of sub-channels, each interfering with only subsequent sub-channels. No further power enhancement is caused as long as ideal precoding with perfect channel side information is assumed.

A duality between the downlink broadcast channel and the uplink multiple access channel (MAC) was recently observed in [6–8]. This duality says that the same rates can be achieved in uplink and downlink under the same power constraint. Hence, the problem of finding the optimal multiuser MIMO transmission strategy in the downlink is equivalent to finding the optimal transmit strategy for the dual uplink problem, which is more tractable. In particular, the MAC sum rate function is concave, thus the sum capacity achieving transmit covariances can be computed by the determinant maximization technique proposed in [9].

However, this approach does not make use of the special analytical structure of the sum rate optimization problem. Given the optimal power allocation, the covariances can be found by the iterative waterfilling technique proposed in [10]. This was exploited in [1], where a dedicated algorithm was proposed that maximizes the sum rate by successively performing power allocation and iterative waterfilling under a sum power constraint. Having found the optimal uplink covariances matrices, the capacity achieving downlink covariances are found with the transformation law proposed in [7].

In this paper we extend these results by providing a necessary and sufficient condition for the optimality of this dedicated strategy. We also characterize the capacity achieving transmission strategy for the low-SNR regime. It turns out that only the user with the largest maximal eigenvalue of the channel covariance matrix is supported. Finally, we discuss the practically relevant case that no Dirty-Paper precoding is performed. In this case the sum rate function does not need to be concave. We illustrate this behavior by numerical simulations.

2. SUM CAPACITY OF THE DUAL MAC

Consider K interfering MIMO links. Each link has n_T transmit antennas and n_R receive antennas. The ith user transmits with a spatial transmit covariance matrix $\boldsymbol{\mathcal{Q}}_i \in \mathbb{C}^{n_T \times n_T}$. We assume uncorrelated noise with covariance $\sigma_n^2 \boldsymbol{I}_{n_R}$. The channel matrix for user i is denoted by $\boldsymbol{H}_i \in \mathbb{C}^{n_R \times n_T}$. Then, the sum rate of the Gaussian MAC with successive decoding of any order is given by

$$f(\boldsymbol{\mathcal{Q}}_1, \ldots, \boldsymbol{\mathcal{Q}}_K) = \log_2 \det\{\boldsymbol{I}_{n_r} + \tfrac{1}{\sigma_n^2} \sum_{k=1}^{K} \boldsymbol{H}_k \boldsymbol{\mathcal{Q}}_k \boldsymbol{H}_k^*\} \ . \quad (1)$$

The transmission powers of the dual MAC are constrained in the same way as the BC. That is, the sum of all powers must be less than a threshold P_{max}. The maximal sum capacity is given by

$$\mathcal{C}_{\text{sum}}(P_{max}) = \max_{\boldsymbol{\mathcal{Q}}_1 \cdots \boldsymbol{\mathcal{Q}}_K} f(\boldsymbol{\mathcal{Q}}_1, \ldots, \boldsymbol{\mathcal{Q}}_K) \quad (2)$$

$$\text{s.t.} \ \ \boldsymbol{\mathcal{Q}}_k \succeq 0 \quad \text{and} \quad \sum_{k=1}^{K} \text{Tr}\{\boldsymbol{\mathcal{Q}}_k\} \le P_{max} \ ,$$

where $\succeq 0$ means positive semidefinite.

By solving problem (2) and applying the transformation law proposed in [7], the capacity achieving BC input covariances can be found. Hence, in the remainder of this paper we can focus on

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solving problem (2) which is easier to handle than direct BC sum rate optimization.

The Lagrangian of problem (2) is given by

$$\mathcal{L}(\mathcal{Q}, \boldsymbol{\Psi}, \mu) = -f(\mathcal{Q}_1, \ldots, \mathcal{Q}_K) - \mu\Big(P_{max} - \sum_{k=1}^{K} \mathrm{Tr}\{\mathcal{Q}_k\}\Big) - \sum_{k=1}^{K} \mathrm{Tr}\{\mathcal{Q}_k \boldsymbol{\Psi}_k\} \ ,$$

where the Lagrangian multipliers $\boldsymbol{\Psi}_k$ are positive semidefinite and μ is non-negative real. The sum rate objective (1) is concave with respect to the transmit covariance matrices $\mathcal{Q}_i$, thus the Karush-Kuhn-Tucker (KKT) conditions [11, 12] are necessary and sufficient for optimality of certain solutions $\mathcal{Q}_1^{opt} \ldots \mathcal{Q}_K^{opt}$. With $\partial\mathcal{L}/\partial\mathcal{Q}_i = 0$, the KKT conditions are

$$\mu \boldsymbol{I} - \boldsymbol{\Psi}_i = \boldsymbol{H}_i^* \big[\sigma_n^2 \boldsymbol{I} + \sum_{k=1}^{K} \boldsymbol{H}_k \mathcal{Q}_k^{opt} \boldsymbol{H}_k^*\big]^{-1} \boldsymbol{H}_i, \quad 1 \leq i \leq K, \tag{3}$$

$$\mathrm{Tr}\{\mathcal{Q}_i^{opt} \boldsymbol{\Psi}_i\} = 0, \qquad 1 \leq i \leq K \tag{4}$$

$$\boldsymbol{\Psi}_i \succeq 0, \qquad 1 \leq i \leq K \tag{5}$$

$$\mathcal{Q}_i^{opt} \succeq 0, \qquad 1 \leq i \leq K \tag{6}$$

$$\mu \geq 0, \tag{7}$$

$$P_{max} - \sum_{k=1}^{K} \mathrm{Tr}(\mathcal{Q}_k^{opt}) \geq 0 \ . \tag{8}$$

3. SUCCESSIVE POWER CONTROL AND ITERATIVE WATERFILLING

Now, we provide necessary and sufficient conditions for when the dedicated algorithm proposed in [1] solves the sum rate maximization problem (2). The algorithm consists of two steps, which are repeated until convergence:

1. For fixed transmission powers, find the optimal covariances by iterative waterfilling [10].
2. For fixed covariances, find the optimal power allocation by the interior point technique proposed in [9].

3.1. Iterative Waterfilling

Define $\mathcal{Q}_i \stackrel{\text{def}}{=} P_i \boldsymbol{Q}_i$, where P_i is the transmission power of the ith user. Thus, the covariance matrix $\boldsymbol{Q}_i$ is normalized such that $\mathrm{Tr}\{\boldsymbol{Q}_i\} = 1$. We define the function $f(\boldsymbol{Q}, P)$ using (1) as

$$f(\boldsymbol{Q}, P) \stackrel{\text{def}}{=} f(P_1' \boldsymbol{Q}_1, \cdots, P_K' \boldsymbol{Q}_K) \ .$$

For fixed transmission powers $P_1' \ldots P_K'$, optimization is performed with respect to $\boldsymbol{Q}$:

$$\{\boldsymbol{Q}_1^{opt}, \ldots, \boldsymbol{Q}_K^{opt}\} = \arg\max_{\boldsymbol{Q}} f(\boldsymbol{Q}, P') \quad \text{s.t.} \ \ \mathrm{Tr}\{\boldsymbol{Q}_k\} = 1, \ \text{and} \ \boldsymbol{Q}_k \succeq 0, \ \forall k. \tag{9}$$

This is done by the iterative waterfilling technique proposed in [10]. The Lagrangian is given by

$$\mathcal{L}_1(\boldsymbol{Q}, \tilde{\boldsymbol{\Psi}}, \tilde{\mu}) = -f(\boldsymbol{Q}, P') - \sum_{k=1}^{K} \tilde{\mu}_k \big(1 - \mathrm{Tr}\{\boldsymbol{Q}_k\}\big) - \sum_{k=1}^{K} \mathrm{Tr}\{\boldsymbol{Q}_k \boldsymbol{\Psi}_k\} \ .$$

With $\partial\mathcal{L}_1/\partial\boldsymbol{Q}_i = 0$ the necessary and sufficient KKT conditions are

$$\tilde{\mu}_i \boldsymbol{I} - \tilde{\boldsymbol{\Psi}}_i = P_i' \boldsymbol{H}_i^* \big[\sigma_n^2 \boldsymbol{I} + \sum_{k=1}^{K} P_k' \boldsymbol{H}_k \boldsymbol{Q}_k^{opt} \boldsymbol{H}_k^*\big]^{-1} \boldsymbol{H}_i, \quad 1 \leq i \leq K \tag{10}$$

$$\mathrm{Tr}\{\boldsymbol{Q}_i^{opt} \tilde{\boldsymbol{\Psi}}_i\} = 0, \quad 1 \leq i \leq K \tag{11}$$

$$\tilde{\boldsymbol{\Psi}}_i \succeq 0, \quad 1 \leq i \leq K \tag{12}$$

$$\boldsymbol{Q}_i^{opt} \succeq 0, \quad 1 \leq i \leq K \tag{13}$$

$$\tilde{\mu}_i \geq 0, \quad 1 \leq i \leq K \tag{14}$$

$$1 - \mathrm{Tr}\{\boldsymbol{Q}_i^{opt}\} \geq 0, \quad 1 \leq i \leq K \ . \tag{15}$$

3.2. Power Allocation

Now, $\boldsymbol{Q}'$ is kept fixed and optimization is performed with respect to $P_1 \ldots P_K$:

$$\{P_1^{opt}, \ldots, P_K^{opt}\} = \arg\max_{P_1 \ldots P_K} f(\boldsymbol{Q}', P) \quad \text{s.t.} \ \sum_{k=1}^{K} P_k \leq P_{max} \ \text{ and } \ P_k \geq 0. \tag{16}$$

The Lagrangian is given by

$$\mathcal{L}_2(P, \hat{\lambda}, \hat{\mu}) = -f(\boldsymbol{Q}', P) - \hat{\mu}\big(P_{max} - \sum_{k=1}^{K} P_k\big) - \sum_{k=1}^{K} P_k \hat{\lambda}_k \ .$$

With $\partial\mathcal{L}_2/\partial P_i = 0$, the KKT conditions are:

$$\hat{\mu} - \hat{\lambda}_i = \mathrm{Tr}\Big\{\big[\sigma_n^2 \boldsymbol{I} + \sum_{k=1}^{K} P_k^{opt} \boldsymbol{H}_k \boldsymbol{Q}_k' \boldsymbol{H}_k^*\big]^{-1} \boldsymbol{H}_i \boldsymbol{Q}_k \boldsymbol{H}_i^*\Big\}, \quad 1 \leq i \leq K \ . \tag{17}$$

$$\sum_{k=1}^{K} P_k^{opt} \hat{\lambda}_k = 0 \tag{18}$$

$$\hat{\lambda}_i \geq 0, \quad 1 \leq i \leq K \tag{19}$$

$$\hat{\mu} \geq 0 \tag{20}$$

$$P_i^{opt} \geq 0, \quad 1 \leq i \leq K \tag{21}$$

$$0 \leq P_{max} - \sum_{k=1}^{K} P_k^{opt} \ . \tag{22}$$

3.3. Optimality of the Iterative Approach

The following theorem provides a necessary and sufficient condition for the optimality of the proposed iterative strategy with respect to the original problem (2). We define the set of active users as

$$\mathcal{I} = \{k \in [1, K] : P_k' > 0\}.$$

Theorem 1. *Suppose that the set of covariance matrices* $\{\boldsymbol{Q}_i'\}$ *solves (9) for given* $\{P_i'\}$ *and* $\{P_i'\}$ *solves (16) for given* $\{\boldsymbol{Q}_i'\}$. *The covariance matrices* $\hat{\boldsymbol{Q}}_k = P_k' \boldsymbol{Q}_k'$ *solve problem (2) if and* *only if there exists a* $\bar{\mu} \geq 0$ *such that*

$$\frac{\tilde{\mu}_k}{P'_k} = \bar{\mu}, \quad k \in \mathcal{I}, \tag{23}$$

$$\bar{\mu}\boldsymbol{I} - \boldsymbol{H}_i^* \big[\sigma_n^2 \boldsymbol{I} + \sum_{k=1}^{K} P'_k \boldsymbol{H}_k \boldsymbol{Q}'_k \boldsymbol{H}_k^*\big]^{-1} \boldsymbol{H}_i \succeq 0, \tag{24}$$

$$i \in [1, K] \backslash \mathcal{I} \; .$$

Proof. Suppose that (23) and (24) are fulfilled. With (10) this implies

$$\boldsymbol{H}_i^* \big[\sigma_n^2 \boldsymbol{I} + \sum_{k=1}^{K} \boldsymbol{H}_k \hat{\boldsymbol{Q}}_k \boldsymbol{H}_k^*\big]^{-1} \boldsymbol{H}_i = \bar{\mu}\boldsymbol{I} - \bar{\boldsymbol{\Psi}}_i, \quad i \in \mathcal{I} \; , \tag{25}$$

where $\bar{\boldsymbol{\Psi}}_i = \tilde{\boldsymbol{\Psi}}_i / P'_i$ is positive semidefinite. With (11) we have $\mathrm{Tr}\{\boldsymbol{Q}'_i \bar{\boldsymbol{\Psi}}_i\} = 0$. For $i \in [1, K] \backslash \mathcal{I}$ we can choose $\boldsymbol{Q}'_i = 0$. Therefore $\mathrm{Tr}\{\boldsymbol{Q}'_i \bar{\boldsymbol{\Psi}}_i\} = 0$ for any positive semidefinite $\bar{\boldsymbol{\Psi}}_i$. Since $\bar{\boldsymbol{\Psi}}_i$ is positive semidefinite, (25) will also be fulfilled for $i \in [1, K] \backslash \mathcal{I}$. This is an immediate consequence of (24). Hence, the KKT conditions (3) and (4) are fulfilled for all k, which implies optimality with respect to problem (2).

To prove the reverse direction, assume that $\hat{\boldsymbol{Q}}_i$ is optimal. Then, there exists a decomposition $\hat{\boldsymbol{Q}}_i = P'_i \boldsymbol{Q}'_i$. The quantities P'_i and $\boldsymbol{Q}'_i$ solve the partial problems (16) and (9), respectively. Otherwise, it would be possible to achieve a sum rate larger than the optimum of the objective $f(\boldsymbol{\mathcal{Q}}_1, \ldots, \boldsymbol{\mathcal{Q}}_K)$, which is a contradiction. From the KKT conditions (3) and (4) immediately follows (23) and (24). ∎

4. CHARACTERIZATION OF THE LOW-SNR OPTIMUM

Now, we characterize the capacity achieving transmit strategy in the low SNR regime.

It can be shown that for low SNR, only one user is active, i.e, one user achieves the optimal sum capacity by transmitting at full power P_{max}. The SNR range in which single-user transmission achieves capacity depends on the channel parameters. Now, an interesting question is: which one is the active user and how can this choice be related to the channel state?

The active user (denoted by index i) transmits at a capacity

$$\mathcal{C}_{\text{sum}}(P_{max}) = \log_2 \det\{\boldsymbol{I} + \tfrac{P_{max}}{\sigma_n^2} \boldsymbol{H}_i \boldsymbol{Q}_i(P_{max}) \boldsymbol{H}_i^*\} \tag{26}$$

$$= \mathrm{Tr} \log_2\{\boldsymbol{I} + \tfrac{P_{max}}{\sigma_n^2} \boldsymbol{H}_i \boldsymbol{Q}_i(P_{max}) \boldsymbol{H}_i^*\} \; , \tag{27}$$

as discussed in [13]. The covariance matrix $\boldsymbol{Q}_i(P_{max})$ denotes the single user waterfilling solution for P_{max}. Decomposing the log function in a Taylor series, we have

$$\mathcal{C}_{\text{sum}}(P_{max}) = \tfrac{\alpha P_{max}}{\sigma_n^2} \mathrm{Tr}\{\boldsymbol{H}_i \boldsymbol{Q}_i(P_{max}) \boldsymbol{H}_i^*\} - R \tag{28}$$

$$\text{with } R = \alpha \sum_{n=2}^{\infty} \tfrac{(-1)^n}{n} (\tfrac{P_{max}}{\sigma_n^2})^n \, \mathrm{Tr}\{\boldsymbol{H}_i \boldsymbol{Q}_i(P_{max}) \boldsymbol{H}_i^*\}^n \; ,$$

where the factor $\alpha = 1/\log_{\mathrm{e}}(2)$ accounts for the fact that this series expansion is defined for the natural logarithm.

The first derivative of the capacity function $\mathcal{C}_{\text{sum}}(P_{max})$ for $P_{max} \to 0$ is given by

$$\lim_{P_{max} \to 0} \frac{\mathcal{C}_{sum}(P_{max}) - \mathcal{C}_{sum}(0)}{P_{max}} = \tfrac{\alpha}{\sigma_n^2} \mathrm{Tr}\{\boldsymbol{H}_i \boldsymbol{Q}_i(0) \boldsymbol{H}_i^*\} \; . \tag{29}$$

This is the slope of the capacity function. Clearly, the active user is the one that maximizes (29). The optimal transmit covariance of the active user is obtained by the waterfilling solution $\boldsymbol{Q}_i(0) = \boldsymbol{V}_i \boldsymbol{\Sigma}_i \boldsymbol{V}_i^*$. The matrix $\boldsymbol{\Sigma}_i = \mathrm{diag}\{\sigma_i^{(1)}, \ldots, \sigma_i^{(n_t)}\}$ fulfills $\mathrm{Tr}\{\boldsymbol{\Sigma}_i\} = 1$. The unitary matrix $\boldsymbol{V}_i$ is found from singular value decomposition $\boldsymbol{H}_i = \boldsymbol{U}_i \boldsymbol{\Lambda}_i^{1/2} \boldsymbol{V}_i^*$. Hence,

$$\mathrm{Tr}\{\boldsymbol{H}_i \boldsymbol{Q}_i(0) \boldsymbol{H}_i^*\} = \mathrm{Tr}\{\boldsymbol{\Lambda}_i \boldsymbol{\Sigma}_i\} = \sum_{l=1}^{n_T} \sigma_i^{(l)} \lambda_i^{(l)} \; , \tag{30}$$

where $\lambda_i^{(1)} \geq \cdots \geq \lambda_i^{(n_T)}$ are the eigenvalues of the channel covariance matrix.

In the low SNR regime with $P_{max} \to 0$, only the maximal eigenvalue is supported by waterfilling. Now, consider two users i and j. For user i assume a maximal eigenvalue with algebraic multiplicity $r \geq 1$, i.e., $\lambda_i^{(1)} = \cdots = \lambda_i^{(r)} > \cdots \geq \lambda_i^{(n_T)}$, the waterfilling solution is

$$\sigma_i^{(l)} = \begin{cases} 1/r, & 1 \leq l \leq r \\ 0, & \text{otherwise} \end{cases} .$$

And for user j assume a maximal eigenvalue with algebraic multiplicity $f \geq 1$, i.e. $\lambda_j^{(1)} = \cdots = \lambda_j^{(f)} > \cdots \geq \lambda_j^{(n_T)}$ with waterfilling solution $\sigma_j^{(l)}$. The term in (30) for user i is greater or equal to the term for user j if and only if $\lambda_i^{(1)} \geq \lambda_j^{(1)}$. Hence, expressions (30) and (29) only depend on the maximal eigenvalue of the channel covariance matrix.

Theorem 2. *For small SNR values, only the user with the largest maximum eigenvalue is supported.*

Remark: The potential received power transmitted over the channel corresponds with the Frobenius norm, thus one might expect that this would be the relevant channel parameter which determines the choice of the active user. However, Theorem 2 shows that the choice of the active user depends only on the maximum eigenvalue of the channel matrices, which is associated with the ℓ_2 norm.

5. SUM CAPACITY WITHOUT PRECODING

So far it was shown that the concavity of the sum rate function (1) allows for an efficient algorithmic solution. However, concavity is only guaranteed as long as decision feedback precoding is used. Otherwise, the sum rate function becomes

$$\hat{f}(\boldsymbol{Q}, P) = K \log_2 \det\Big\{\sigma_n^2 \boldsymbol{I} + \sum_{k=1}^{K} P_k \boldsymbol{H}_k \boldsymbol{Q}_k \boldsymbol{H}_k^*\Big\} - \sum_{i=1}^{K} \log_2 \det\Big\{\sigma_n^2 \boldsymbol{I} + \sum_{\substack{k=1 \\ k \neq i}}^{K} P_k \boldsymbol{H}_k \boldsymbol{Q}_k \boldsymbol{H}_k^*\Big\} \; .$$

Observe that this is the sum of a concave and a convex function. The result does not need not be convex neither concave, as illustrated in Fig. 1.b).

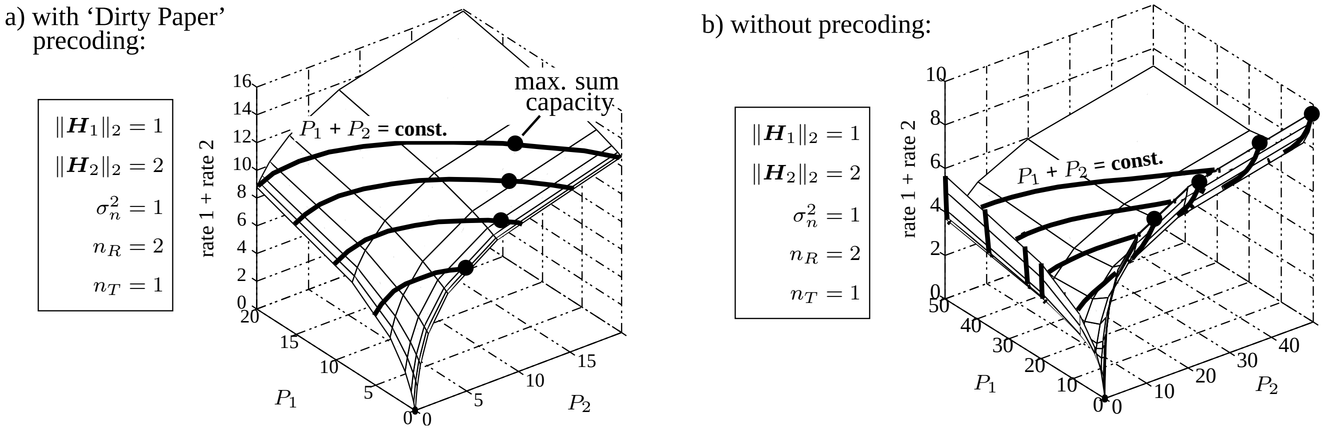

Fig. 1. Sum rate of the dual MAC vs. transmission powers P_1 and P_2. The optimal sum capacity is given as the maximum sum rate under a sum power constraint $P_1 + P_2 \leq P_{max}$. a) With ‘Dirty Paper’ precoding [3, 4], the capacity function is always concave. b) Without precoding, however, concavity is not guaranteed.

6. CONCLUSIONS

In this paper we study the sum capacity of the Gaussian MIMO broadcast channel. We make use of the duality between broadcast and multiple access channel, which was recently described in [7]. The dual problem has a concave sum rate function and is therefore much easier to handle than the original problem. This was already exploited in [1], where an algorithm was proposed that approaches the optimum sum capacity by successively performing iterative waterfilling and power control.

In this paper we have extended these results by providing necessary and sufficient conditions for optimal convergence. Numerical simulations indicate that these conditions are always fulfilled, except for the case when one user is switched off prematurely during the iteration process. However, this can easily be avoided, e.g. by adding an additional barrier term.

The main advantage of this dedicated algorithm is its low computational complexity, as compared to direct optimization via determinant maximization [9]. Complexity gains are made possible by exploiting the specific analytical structure of the given problem, namely the characterization of the input covariances via the iterative waterfilling solution.

While this algorithm achieves the sum capacity for any SNR, the problem is reduced in the low SNR regime. It has been shown that in this case the optimal sum capacity is achieved by transmitting all the power in the direction of the user with the largest maximal eigenvalue (ℓ_2 norm of the channel matrix).

Finally, we study the sum capacity without precoding. We show that the sum capacity is neither a concave nor convex function of the transmit covariance matrices. This property is illustrated by numerical simulations. The sum capacity optimization without precoding is an interesting topic for further research.

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