



# TESTS FOR ASSESSING MULTIVARIATE NORMALITY AND THE COVARIANCE STRUCTURE OF MIMO DATA\*

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## ABSTRACT

Measurements taken at the campus of Brigham Young University (BYU) are used to investigate the statistical properties of the indoor MIMO channel. Two statistical tests, Royston's and Henze-Zirkler's, are applied to the MIMO data to assess whether the data belongs to a multivariate normal distribution or not. The possibility of modeling the covariance matrix as a Kronecker product of the correlations at the transmitter and receiver are also investigated by deriving a likelihood ratio test. It is found that small MIMO systems such as  $2 \times 2$  can be considered normally distributed and can also be approximated with a Kronecker structure. Larger systems, on the other hand, show evidence of strong non-normality and is not well modeled using a Kronecker product. However, for short measurement segments, these distributions can be used for approximate channel capacity calculations.

## 1. INTRODUCTION

Systems with antenna arrays at both the transmitter and receiver, so called Multiple-Input Multiple-Output (MIMO) systems, have recently been shown to be capable of providing very high bit rates [1, 2]. These rates are achieved by utilizing the spatial domain to a larger extent than previously. Since no additional bandwidth is required, MIMO systems have attracted considerable attention. Several MIMO measurement campaigns have recently been reported in [3, 4, 5, 6] where the measured channel capacity in different environments has been investigated. However, the dominating assumption in system analysis of complex normally distributed channel coefficients has not been investigated in any detail. Results regarding the marginal statistics have been reported in [5, 6] where it was found that the channel coefficients were Rayleigh distributed in Non Line Of Sight (NLOS) and Rice distributed in Line Of Sight (LOS). Since the phase was found to be uniformly distributed, a complex Gaussian distribution for the channel coefficients was found to be a reasonable assumption.

However, the fact that each coefficient has a univariate normal distribution does not imply that the channel matrix must belong to a multivariate normal distribution. In fact it will be shown by applying two tests for MultiVariate Normality (MVN) to measured MIMO channels that larger MIMO systems show strong evidence

of non-normality but each individual coefficient is close to a complex normal distribution.

Furthermore, the structure of the covariance matrix is also investigated. It has been suggested that the correlation between two subchannels can be modeled by a product of the correlations seen by the transmitter and receiver using a Kronecker product. This has been studied previously in terms of predicted channel capacity [6]. Here, however, a likelihood ratio test is derived to assess the covariance structure. Finally, the results of the different tests are related to the channel capacities found under the different assumptions.

## 2. TESTS FOR MULTIVARIATE NORMALITY

There are many tests for MVN in the statistical literature [7]. Unfortunately, there is no known uniformly most powerful test and it is recommended to perform several test to assess MVN. In this section, two tests that can be applied to measured MIMO data will be described. These tests have been found [7] to have good overall power against alternatives to normality.

### 2.1. Royston's $H$ test

Royston's  $H$  test [8] is a multivariate extension of a popular test for UniVariate Normality (UVN), the Shapiro-Wilk  $W$  test [9]. The Shapiro-Wilk test is generally considered to be an excellent test for UVN [7]. Let  $W_j$  denote the value of the Shapiro-Wilk statistic for the  $j$ th variable in a  $p$ -variate distribution. Then, define

$$R_j = \left\{ \Phi^{-1} \left[ \frac{1}{2} \Phi \left\{ - \left( (1 - W_j)^\lambda - \mu \right) / \sigma \right\} \right] \right\}^2, \quad (1)$$

where  $\lambda$ ,  $\mu$ , and  $\sigma$  are calculated from polynomial approximations given in [9] and  $\Phi(\cdot)$  denotes the standard normal cdf. Now if the data is MVN,  $H = \xi \sum R_j / p$  is approximately  $\chi_\xi^2$  distributed, where

$$\hat{\xi} = p / [1 + (p - 1)\bar{c}], \quad (2)$$

where  $\bar{c}$  is an estimate of the average correlation among the  $R_j$ 's [8]. This  $\chi_\xi^2$  distribution is used to obtain critical values for the test. Royston's  $H$  test was in [7] found to have good power against many different alternative distributions.

### 2.2. Henze-Zirkler's Test

Another MVN known for good power [7] is the Henze-Zirkler test [10] that is based on the empirical characteristic function. An appealing property of this test is that it is a consistent test. Let  $\mathbf{x}_j$

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denote the  $j$ th vector sample out of  $n$  samples of a  $p$  variate distribution. Then, the test statistic is formed as

$$T = \frac{1}{n} \sum_{j,k=1}^n \exp\left(-\frac{\beta^2}{2} |\mathbf{y}_j - \mathbf{y}_k|^2\right) - 2(1 + \beta^2)^{-\frac{d}{2}} \times \sum_{j=1}^n \exp\left(-\frac{\beta^2}{2(1 + \beta^2)} |\mathbf{y}_j|^2\right) + n(1 + 2\beta^2)^{-\frac{d}{2}}, \quad (3)$$

where  $|\mathbf{y}_j - \mathbf{y}_k|^2 = (\mathbf{x}_j - \mathbf{x}_k)^H \hat{\mathbf{R}}^{-1} (\mathbf{x}_j - \mathbf{x}_k)$ ,  $|\mathbf{y}_j|^2 = (\mathbf{x}_j - \bar{\mathbf{x}})^H \hat{\mathbf{R}}^{-1} (\mathbf{x}_j - \bar{\mathbf{x}})$ , and  $\bar{\mathbf{x}}$ ,  $\hat{\mathbf{R}}$  denote the sample mean vector and covariance matrix, respectively. The parameter  $\beta$  in (3) represents a smoothing parameter [10] and in this paper  $\beta = 0.5$  will be used. If the data is MVN, the test statistic  $T$  is approximately lognormally distributed with

$$E[T] = 1 - (1 + 2\beta^2)^{-\frac{d}{2}} \left[ 1 + \frac{d\beta^2}{1+2\beta^2} + \frac{d(d+2)\beta^4}{2(1+2\beta^2)^2} \right]$$

$$Var[T] = 2(1 + 4\beta^2)^{-\frac{d}{2}} + 2(1 + 2\beta^2)^{-d} \left[ 1 + \frac{2d\beta^4}{(1+2\beta^2)^2} + \frac{3d(d+2)\beta^8}{4(1+2\beta^2)^4} \right] - 4w^{-\frac{d}{2}} \left[ 1 + \frac{3d\beta^4}{2w} + \frac{d(d+2)\beta^8}{2w^2} \right], \quad (4)$$

where  $w = (1 + \beta^2)(1 + 3\beta^2)$ . This result is used to find the critical values of the test.

### 3. A TEST FOR KRONECKER STRUCTURE

To examine if the covariance matrix can be modeled as a Kronecker product between the transmitter and receiver covariance matrices, a likelihood ratio test was derived. The null hypothesis is a MVN distribution with a Kronecker covariance matrix while the general alternative is a MVN with arbitrary covariance

$$H_0 : \mathbf{x} \in \mathcal{N}(\boldsymbol{\mu}, \mathbf{R}_1 \otimes \mathbf{R}_2)$$

$$H_1 : \mathbf{x} \in \mathcal{N}(\boldsymbol{\mu}, \mathbf{R}). \quad (5)$$

Here,  $\mathbf{R}$  is an  $p \times p$  matrix while  $\mathbf{R}_1$  and  $\mathbf{R}_2$  are  $p_1 \times p_1$  and  $p_2 \times p_2$  with  $p_1 p_2 = p$ . Using the standard matrix formulation of the multivariate complex Gaussian pdf, the likelihood ratio  $\Lambda$  can be written as

$$\Lambda = \frac{\left| \hat{\mathbf{R}}_1 \otimes \hat{\mathbf{R}}_2 \right|^{-\frac{n}{2}} e^{-\frac{1}{2} \sum_{r=1}^n (\mathbf{x}_r - \hat{\boldsymbol{\mu}})^H (\hat{\mathbf{R}}_1 \otimes \hat{\mathbf{R}}_2)^{-1} (\mathbf{x}_r - \hat{\boldsymbol{\mu}})}}{\left| \hat{\mathbf{R}} \right|^{-\frac{n}{2}} e^{-\frac{1}{2} \sum_{r=1}^n (\mathbf{x}_r - \hat{\boldsymbol{\mu}})^H \hat{\mathbf{R}}^{-1} (\mathbf{x}_r - \hat{\boldsymbol{\mu}})}}. \quad (6)$$

Note that for the test to be a true likelihood test, Maximum Likelihood (ML) estimates of all the estimated quantities  $(\hat{\cdot})$  should be used. Finding the ML estimates of the sample mean and covariance is straightforward but determining the ML estimates of  $\mathbf{R}_1$  and  $\mathbf{R}_2$  is more difficult. Now, for MIMO data, the vector  $\mathbf{x}$  represents the stacked data, i.e.  $\mathbf{x} = \text{vec}(\mathbf{H})$  where the  $n_r \times n_t$  matrix  $\mathbf{H}$  is the channel matrix. In this case,  $\mathbf{R}_1$  corresponds to the transmitter covariance  $\mathbf{R}_t$  and  $\mathbf{R}_2$  the receiver covariance  $\mathbf{R}_r$ . Hence, straightforward estimates of the transmitter and receiver covariances may be used

$$\hat{\mathbf{R}}_1 = \hat{\mathbf{R}}_t = \frac{1}{np_2} \sum_{k=1}^n [\mathbf{H}(k) - \bar{\mathbf{H}}(k)]^H [\mathbf{H}(k) - \bar{\mathbf{H}}(k)] \quad (7)$$

$$\hat{\mathbf{R}}_2 = \hat{\mathbf{R}}_r = \frac{1}{np_1} \sum_{k=1}^n [\mathbf{H}(k) - \bar{\mathbf{H}}(k)] [\mathbf{H}(k) - \bar{\mathbf{H}}(k)]^H. \quad (8)$$

In [6] it was found that the estimators in (7) and (8) were close to a least squares approach  $\{\hat{\mathbf{R}}_1, \hat{\mathbf{R}}_2\} = \min_{\mathbf{R}_1, \mathbf{R}_2} \|\hat{\mathbf{R}} - \mathbf{R}_1 \otimes \mathbf{R}_2\|_F^2$  where  $\|\cdot\|_F$  denotes the Frobenius norm. Here, the estimators in (7) and (8) will be used since the least squares approach requires a multiplication involving a large matrix. For example, a  $7 \times 7$  MIMO systems results in a matrix product involving a  $7^4 \times 7^4$  matrix. Taking the logarithm of (6) and using that  $\text{Tr}[\mathbf{AB}] = \text{Tr}[\mathbf{BA}]$  and that  $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$

$$-2 \log \Lambda = n \{\text{Tr} \mathbf{M} - \log |\mathbf{M}| - p\}, \quad (9)$$

where  $\mathbf{M} = (\hat{\mathbf{R}}_1 \otimes \hat{\mathbf{R}}_2)^{-1} \hat{\mathbf{R}}$ . No attempt to derive an expression for the distribution of the statistic under the null hypothesis was made. Instead, Monte-Carlo simulations provided the critical values for the test.

## 4. MEASUREMENT SETUP

A narrowband custom made MIMO communications system designed and built at Brigham Young University (BYU) in Utah was used to collect measurements. The system was equipped with ten monopoles forming a uniform circular array at each end. However, since the elements were mounted over a ground plane the monopoles behave as dipoles and essentially have the same radiation patterns as dipoles. Furthermore, the elements were positioned in a circle with radius 0.86 wavelengths that approximately gives an element separation of a half wavelength. The operating frequency was 2.43GHz. For a detailed description of the measurement equipment, see [5].

Measurements were collected within the Clyde building at the BYU campus. A measurement path in NLOS was chosen since it was expected that it would represent an environment with reasonably stationary statistics. The sampling rate was set to 2.5ms in order to get an oversampled channel with many samples per wavelength. Over a measurement length of 42m, 10000 MIMO samples were collected, corresponding to about 30 samples per wavelength.

## 5. MEASUREMENT RESULTS

### 5.1. Multivariate Statistics

Since the measurements were collected over many wavelengths, normalization of the data is necessary. A running mean filter is one possibility but requires choosing a suitable averaging length of the filter. Instead, the average of the magnitude of all channel coefficient is used to normalize the channel matrix at each location. That represents a spatial average over the area of the array which is about  $1.7\lambda \times 1.7\lambda$ . This is found to yield a reasonable stable power compensation over the measurement path. In order to avoid a high correlation between neighboring samples, the data was downsampled by using only one sample per wavelength. This yielded a maximum correlation between neighboring samples of 0.25.

The power is not the only channel characteristic changing along the path. The correlation between the different coefficients also changes along the path. Therefore, the MVN tests from Section 2 are not applied to the entire measurement path but instead to subsections of the path of length  $40\lambda$ . In order to build statistics, the MVN tests were applied to 100 subsections of length  $40\lambda$  which were obtained by sliding a window of  $40\lambda$  along the measurement path.

		$2 \times 2$	$3 \times 3$	$4 \times 4$	$5 \times 5$	$6 \times 6$
$R-H$	Real	1%	2%	5%	1%	0%
	Imag	2%	3%	3%	6%	8%
$H-Z$	Real	10%	14%	32%	100%	100%
	Imag	2%	5%	25%	100%	100%

**Table 1.** The rejection rates of Royston's and Henze-Zirkler's MVN tests applied to the real and imaginary part of the channel averaged over the subsections for different array sizes.

First, the univariate properties of the data were investigated since a random vector can only belong to a MVN distribution if every linear combination of the component variables has a univariate normal distribution. Here, Shapiro-Wilk's test and the univariate version of Henze-Zirkler's test were applied using a significance level of 0.05. The tests were applied to both the real part and the imaginary part which were found to be approximately uncorrelated. By averaging over all subsections and all channel coefficients  $\mathbf{H}_{ij}$   $i, j = 1, 2, \dots, p$ , it was found that the average rejection rate was close to the significance level. However, by only averaging along the different subsections, it was found that about 10 of the channel coefficients had a rejection rate of 10%-15%. Hence, there is some evidence of a mild deviation from UVN.

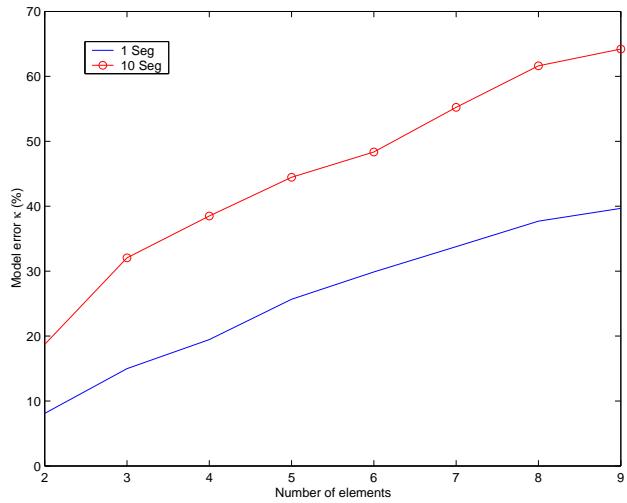
More evidence of non-normality can be obtained by applying the MVN test to the same data since these tests will detect deviation in the multivariate structure. The results from applying Royston's and Henze-Zirkler's tests to the real and imaginary parts of the channel, averaged over the subsections, are shown in Table 1. It is clear that the H-Z test is detecting evidence of non-normality for larger MIMO systems while the R-H test does not. There are two plausible explanations for this: The power of the R-H test is unknown for larger number of variates; The H-Z test with  $\beta = 0.5$  is known to be powerful against distributions with heavy tails which may be the case of the MIMO data. However, for both tests no significant evidence against non-normality was found against  $2 \times 2$  and  $3 \times 3$  systems. The  $4 \times 4$  system has more evidence of non-normality since the H-Z has a fairly high rejection rate while MIMO systems with five elements and above can be considered non-normal in this measurement set.

## 5.2. Kronecker Structure

The test for Kronecker structure derived in Section 3 was applied to the same data as the multivariate tests. In this case, no sliding window was used since the Kronecker test already includes averaging because it is based on estimated covariance matrices. Instead, the entire measurement path was broken into ten equal length segments to which the test was applied. To investigate the possible covariance changes along the path, the test was also applied to the entire path.

The significance levels of the test statistic for the entire path for the  $2 \times 2$ ,  $3 \times 3$ , and  $4 \times 4$  were 0.08, 0.23, 0.68. These values are well below the 5% threshold (0.95). Thus, for these array sizes a Kronecker structure can not be rejected. The remaining larger MIMO systems had larger values either close to or equal to unity and can therefore be rejected. For the ten subsegments case, only the smallest MIMO system, the  $2 \times 2$ , had values less than unity. For the  $2 \times 2$  system, five subsegments had significance levels below 0.95 and hence could not be rejected.

It is clear that the Kronecker structure does not describe the structure of larger MIMO systems well. Another interesting ob-



**Fig. 1.** Model error  $\kappa$  for both the ten subsegment case and the one segment entire measurement case.

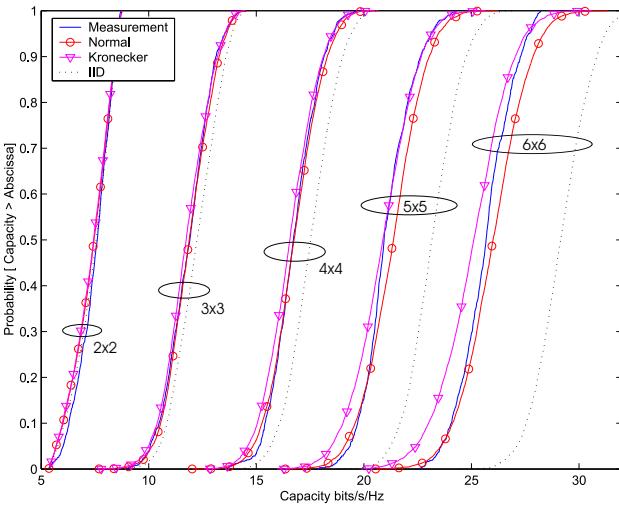
servation is that it appears that over the entire path, the Kronecker structure is more likely to not be rejected than the subsegment case. A plausible explanation for this is that the more the distribution characteristics change over the test segment, the more the covariance estimate will focus on the stable part of the covariance due to the structure of the antennas. Therefore, it is suspected that for measurement campaigns with a wide variety of measurement locations, the Kronecker structure will fit the overall data better than at the individual locations. To summarize, the test results indicates that the distribution of  $2 \times 2$  systems may exhibit a Kronecker structure but larger systems are not described accurately using that structure.

The discrepancy between the general covariance matrix and the Kronecker approximation can be further examined by calculating a model error defined as

$$\kappa = \frac{\|\hat{\mathbf{R}} - \hat{\mathbf{R}}_t \otimes \hat{\mathbf{R}}_r\|_F}{\|\hat{\mathbf{R}}\|_F}. \quad (10)$$

The model error for both the ten subsegment case and the one segment entire measurement case is shown in Figure 1. For both cases, the errors are large but the case when the measurement is divided into ten subsegments has almost twice the error than the case of one segment constituting the entire sequence. This is again an indication that the Kronecker structure not works well for systems above  $2 \times 2$  and that the covariance structure changes along the measurement path.

Previous work on using the Kronecker structure used the channel capacity as a performance measure to determine how well the data fits the structure [6]. Furthermore, results were only presented for  $2 \times 2$  and  $3 \times 3$  systems. In Figure 2, channel capacities for array sizes between two and six are shown for the case of ten sub-segments. The channel capacity was calculated assuming no channel knowledge at the transmitter [1, 2] and using an SNR of 10dB. For this case, both the general MVN and the Kronecker approximations actually are capable of reproducing the channel capacity at a reasonable accuracy. It can also be noted that the capacity loss relative to the upper bound of uncorrelated i.i.d. complex normal distributed coefficients increases with the number of elements.



**Fig. 2.** Channel capacities for array sizes between two and six for the case of ten subsegments.

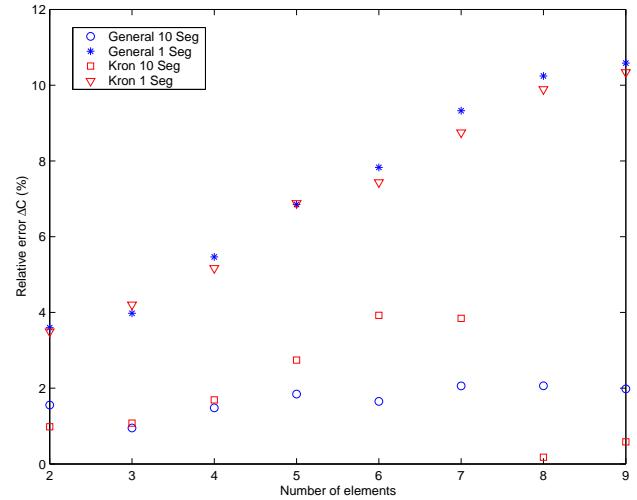
Hence, for this set of measurements, a MVN distribution may not model the multivariate distribution very well but is still capable of modeling the scalar channel capacity.

In Figure 3, the relative error of the channel capacity  $\Delta C = |C_{\text{data}} - \hat{C}|/C_{\text{data}}$  is shown. Here,  $\hat{C}$  denotes the capacity calculated using either the general MVN distribution or the Kronecker approximation. The error is smaller for the case of ten subsegments than for the entire sequence. This indicates that the covariance properties of the channel may change along the path, as discussed above. Hence, a closer fit in capacity can be obtained by calculating several covariances along a measurement path. Therefore, for shorter segments of data, a MVN can be used to for capacity comparisons with errors at about 2%. A Kronecker structure can also be used with slightly higher errors. However, for longer segments, the error exceeds 10% for larger array sizes using either a general or Kronecker structure.

## 6. CONCLUSIONS

Measurements taken at the campus of Brigham Young University (BYU) were used to investigate the statistical properties of the indoor MIMO channel. Two statistical tests, Royston's and Henze-Zirkler's, were applied to the MIMO data to assess whether the data belongs to a multivariate normal distribution or not. The possibility of modeling the covariance matrix as a Kronecker product of the correlations at the transmitter and receiver was also investigated by deriving a likelihood ratio test.

It was found that smaller systems such as  $2 \times 2$  and  $3 \times 3$  can be considered reasonably normal but large MIMO systems show strong evidence of non-normality. Furthermore, it is only these small array sizes that can be approximated with a Kronecker structure. Surprisingly, it was also found that although the MVN may not describe the distribution accurately, it can still be used to simulate channel capacity with a relative error of a few percent. However, that is under the condition that the channel properties vary only slightly over the measurement segment.



**Fig. 3.** The relative error of the channel capacity  $\Delta C$  using a general MVN and a Kronecker approximation for the ten segments and one segment cases.

## 7. REFERENCES

- [1] G.J. Foschini and M.J. Gans, "On Limits of Wireless Communications in a Fading Environment when Using Multiple Antennas," *Wireless Personal Communications*, vol. 6, pp. 311–335, March 1998.
- [2] E. Telatar, "Capacity of Multi-Antenna Gaussian Channels," Tech. Rep., AT&T Bell Labs, June 1995.
- [3] J.P. Kermoal, L. Schumacher, P.E. Mogensen, and K.I. Pedersen, "Experimental Investigation of Correlation Properties of MIMO Radio Channels for Indoor Picocell Scenarios," in *Proc. IEEE VTC 2000 Fall*, Boston, MA, September 2000.
- [4] A.F. Molisch, M. Steinbauer, M. Toeltsch, E. Bonek, and R.S. Thomä, "Capacity of MIMO Systems Based on Measured Wireless Channels," *IEEE Journal on Selected Areas in Communications*, vol. 20, no. 3, pp. 561–569, April 2002.
- [5] J.W. Wallace and M.A. Jensen, "Measured Characteristics of the MIMO Wireless Channel," in *Proc. IEEE VTC 01 Fall*, Atlantic City, NJ, USA, October 2001, pp. 2038–2042.
- [6] K. Yu, M. Bengtsson, B. Ottersten, D. McNamara, P. Karlsson, and M. Beach, "Second Order Statistics of NLOS Indoor MIMO Channels Based on 5.2GHz Measurements," in *Proc. IEEE Globecom 2001*, San Antonio, TX, November 2001, vol. 1, pp. 156–160.
- [7] C.J. Mecklin, *A Comparison of the Power of Classical and Newer Tests of Multivariate Normality*, Ph.D. thesis, University of Northern Colorado, 2000.
- [8] J.P. Royston, "Some Techniques for Assessing Multivariate Normality Based on the Shapiro-Wilk  $W$ ," *Applied Statistics*, vol. 32, no. 2, pp. 121–133, 1983.
- [9] J.P. Royston, "An Extension of Shapiro and Wilk's  $W$  Test for Normality to Large Samples," *Applied Statistics*, vol. 31, no. 2, pp. 115–124, 1982.
- [10] N. Henze and B. Zirkler, "A Class of Invariant Consistent Tests for Multivariate Normality," *Commun. Statist.-Theor. Meth.*, vol. 19, no. 10, pp. 3595–3618, 1990.