LOW COMPLEXITY LINEAR EQUALIZERS WITH MAXIMUM MULTIPATH DIVERSITY FOR ZERO-PADDED TRANSMISSIONS

Cihan Tepedelenlioğlu *

Department of Electrical Engineering
Telecommunication Research Center
Arizona State University
cihan@asu.edu

ABSTRACT

In wireless communications the fading multipath channel attenuates and distorts the transmitted signal. This makes equalization of the frequency selective channel of utmost importance. To exploit the full diversity provided by the multipath channel, maximum likelihood (ML) decoding is usually employed, which is computationally complex. In this paper we show that a specific *linear* zero-forcing equalizer is capable of benefitting from *maximum multipath diversity* in systems where the transmitted blocks are separated by zero guard intervals of length greater than the channel length (the well-known trailing zeros approach [6]). Furthermore, we exploit the banded Toeplitz structure of the channel matrix to reduce the complexity of the equalization process, and quantify the reduction in complexity. Simulations corroborate our results.

1. INTRODUCTION AND SYSTEM MODEL

In wireless communications multipath propagation causes the transmitted signal to be distorted due to the superposition of delayed and attenuated version of the same signal. In a digital communication system, this effect can be captured through a discrete-time equivalent tapped delay line model that relates the matched filtered samples of the received signal $x[n]$, and the transmitted symbols $s[n]$ through a well-known convolution relationship [2]: $x[n] = \sum_{l=0}^{L} h[l]s[n-l]$ where $h[l]$ is the channel impulse response which represents the convolution of the transmitter pulse, the physical channel and the matched (to the transmitter pulse) filter, sampled at multiples of the baud rate. It is well-known that by blocking the information symbols into blocks of length N, and appending $\bar{L} \geq L$ zeros at the end of each block we obtain the following matrix-vector relationship between the blocked input and blocked output [6]:

$$\mathbf{x}[i] = \mathbf{H}\mathbf{s}[i] + \mathbf{v}[i] \ , \tag{1}$$

*This work was supported with the NSF CAREER grant No. CCR-0133841 and the State of Arizona with the IT-301 initiative.

where $\mathbf{x}[i] := [x[iP], \ldots, x[iP+P-1]]^T$, the white Gaussian noise $\mathbf{v}[i] := [v[iP], \ldots, v[iP+P-1]]^T$, satisfies $E[\mathbf{v}[i]\mathbf{v}[i]^H] = \sigma^2\mathbf{I}$, $\mathbf{s}[i] := [s[iN], \ldots, s[iN+N-1]]^T$, and $\mathbf{H}$ is a $P \times N$ banded Toeplitz matrix whose first column is given by $[h[0] \ldots h[L]]^T$. In other words, the (p, n) element of $\mathbf{H}$ is given by $[\mathbf{H}]_{p,n} = h[p-n]$, if $p \geq n$ and $[\mathbf{H}]_{p,n} = 0$ if $p < n$. We also note that $P = N + \bar{L}$.

It is important to notice that, due to the banded Toeplitz structure of $\mathbf{H}$, there always exists a $N \times P$ left inverse $\mathbf{G}$ such that $\mathbf{GH} = \mathbf{I}$ [8] (unless of course $h[l] = 0 \; \forall l$).

The following decomposition of $\mathbf{H}$ will be useful in constructing a zero-forcing equalizer and can easily be verified:

$$\mathbf{H} = \mathbf{F}^H\mathbf{D}\mathbf{\Theta} \ , \tag{2}$$

where $\mathbf{F}$ is a $P \times P$ DFT matrix with (k, l) element $[\mathbf{F}]_{k,l} = N^{-1/2}e^{-j2\pi kl/P}$, $\mathbf{D}$ is a $P \times P$ diagonal matrix whose (p, p) element is given by $[\mathbf{D}]_{p,p} := H(e^{-j2\pi p/P}) = \sum_{l=0}^{L} h[l]e^{-2\pi pl/P}$, and $\mathbf{\Theta}$ consists of the first N columns of $\mathbf{F}$. We see that (2) provides a channel that is very similar to that seen in OFDM systems, with the major difference that the zero-padding approach does not suffer from peak to average ratio problems. Let $\mathbf{h} := [h[0] \ldots h[L]]^T$, be the vector consisting of the $(L+1)$ channel taps. In [8] it is shown that if the covariance matrix of the channel taps $\mathbf{R}_h := E[\mathbf{h}\mathbf{h}^H]$ is full rank, then it is possible to obtain maximal (i.e., $(L+1)$ order) multipath diversity with using *maximum likelihood decoding* at the receiver. We prove in the Section 2 that maximum multipath diversity (MMD) is possible even with *linear* block equalization. In Section 3 we quantify the equalizer complexity; Section 4 illustrates our results with simulations and concludes the paper.

2. PAIRWISE ERROR PROBABILITY ANALYSIS

In this section we derive the pairwise error probability (PEP) for the zero-padded transmission with the assumption that the decoding process is composed of zero-forcing linear equalization followed by elementwise quantization. In what follows, we drop the block index i in (1), since we will be analyzing block-by-block decoding. Under the assumption of linear, zero-forcing equalization (1) becomes [1]:

$$\hat{\mathbf{s}} = \mathbf{G}\mathbf{x} = \mathbf{s} + \mathbf{G}\mathbf{v} = \mathbf{s} + \mathbf{n} \ , \tag{3}$$

where $\mathbf{G}$ denotes any left inverse of $\mathbf{H}$ (not necessarily the unique pseudo-inverse), and $\mathbf{n} := \mathbf{G}\mathbf{v}$. For the PEP analysis, we will be looking at the probability that a block of symbols $\mathbf{s}_j$ is sent but another block $\mathbf{s}_k \neq \mathbf{s}_j$ is received. For this purpose, following [3, pp. 160] let us define $d_{jk} = \parallel \mathbf{s}_k - \mathbf{s}_j \parallel$ to be the distance between $\mathbf{s}_j$ and $\mathbf{s}_k$, $\mathbf{e}_{jk} = (\mathbf{s}_j - \mathbf{s}_k)/ \parallel \mathbf{s}_k - \mathbf{s}_j \parallel$ a unit vector representing the normalized difference, and $z := \mathbf{n}^H \mathbf{e}_{jk}$. Then the pairwise error probability that $\mathbf{s}_j$ is transmitted and incorrectly decoded as $\mathbf{s}_k$ can be expressed as:

$$P\left[\mathbf{s}_j \rightarrow \mathbf{s}_k | \mathbf{h}\right] = P\left[z \geq \frac{d_{jk}}{2}\right] \ . \tag{4}$$

Notice that z is a Gaussian random variable with mean zero and variance given by

$$E|z|^2 = E|\mathbf{n}^H \mathbf{e}_{jk}|^2 = \sigma^2 \parallel \mathbf{G}^H \mathbf{e}_{jk} \parallel^2 \ . \tag{5}$$

Hence, the conditional PEP, that depends on the channel coefficients $\mathbf{h}$ can be expressed as:

$$P\left[\mathbf{s}_j \rightarrow \mathbf{s}_k | \mathbf{h}\right] = Q\left(\frac{d_{jk}}{\sqrt{4\sigma^2 \parallel \mathbf{G}^H \mathbf{e}_{jk} \parallel^2}}\right) \ , \tag{6}$$

where $Q(\cdot)$ is the error function. Assuming that $\mathbf{h}$ is a Gaussian vector with a full rank covariance matrix, we now prove that for a specific left inverse of $\mathbf{H}$ (which we will soon define), the PEP in (6) averaged over the distribution of $\mathbf{h}$ will yield $(L+1)$ order multipath diversity.

Toward this end, let $\kappa := \{p_1, \ldots, p_{\bar{L}}\}$ be the set of indices corresponding to the smallest $\bar{L}$ channel responses so that if $p \in \kappa$ and $q \notin \kappa$ $|H(e^{-j2\pi p/P})| \leq |H(e^{-j2\pi q/P})|$, and the indices in κ are ordered so that $|H(e^{-j2\pi p_1/P})| \leq \ldots \leq |H(e^{-j2\pi p_{\bar{L}}/P})|$. Notice that because the channel can have at most L zeros, if $p \notin \kappa$, then $|H(e^{-j2\pi p/P})| > 0$.

Now we would like to introduce a specific zero-forcing equalizer to $\mathbf{H}$. Let $\mathbf{D}_\kappa$ be a diagonal matrix that can be obtained by the following definition: $[\mathbf{D}_\kappa]_{p,p} = 1/H(e^{-j2\pi p/P})$ if $p \notin \kappa$ and $[\mathbf{D}_\kappa]_{p,p} = 0$ if $p \in \kappa$. Please note that the definition of κ insures that the denominator in the definition of $\mathbf{D}_\kappa$ is nonzero. We also define $\mathbf{\Theta}_\kappa$ following a three step procedure: (i) remove all rows of $\mathbf{\Theta}$ that has indices belonging to κ to obtain an $N \times N$ matrix $\mathbf{A}_\kappa$; (ii) Compute the $\mathbf{A}_\kappa^{-1}$ (note that the inverse exists because of the Vandermonde structure of $\mathbf{A}_\kappa$ which is inherited from $\mathbf{\Theta}$); (iii) Insert $\bar{L}$ zero columns to $\mathbf{A}_\kappa^{-1}$ with columns that have indices belonging to κ to obtain an $N \times P$ matrix $\mathbf{\Theta}_\kappa$. We now define *a* left inverse for $\mathbf{H}$ as follows:

$$\mathbf{G} := \mathbf{\Theta}_\kappa \mathbf{D}_\kappa \mathbf{F} \ . \tag{7}$$

[1] we assume that the channel is known throughout the paper

It is now straightforward to verify that $\mathbf{G}\mathbf{H} = \mathbf{I}$. It is important to stress however, that even though $\mathbf{H}$ has infinitely many left inverses, $\mathbf{G}$, as defined in (7) is unique, and is not necessarily the pseudoinverse of $\mathbf{H}$. The reason we consider $\mathbf{G}$ in (7) as opposed to the pseudoinverse is because the dependence of $\mathbf{G}$ on $\mathbf{h}$ is very explicit, and this will enable us to average (6) with respect to $\mathbf{h}$. In fact, in [5] we show that the pseudoinverse of $\mathbf{H}$ can also be shown to offer MMD, in a similar context where linear precoding together with OFDM is used.

In order to calculate the average (over the channel) PEP, we need to average (6) with respect to the distribution of $\mathbf{h}$. We first observe the well-known fact that the PEP can be upper bounded as follows:

$$Q\left(\frac{d_{jk}}{\sqrt{4\sigma^2 \parallel \mathbf{G}^H \mathbf{e}_{jk} \parallel^2}}\right) \leq \frac{1}{2}\exp\left(\frac{-d_{jk}^2}{8\sigma^2 \parallel \mathbf{G}^H \mathbf{e}_{jk} \parallel^2}\right) \ . \tag{8}$$

We now state our main theorem establishing that linear block equalization offers MMD.

Theorem: *With* $\mathbf{G}$ *defined as in (7) it is possible to upper bound the right hand side of (8) with an expression of the form* $\exp(-\mathrm{SNR} \parallel \mathbf{h} \parallel^2 K)$, *where* K *is a constant independent of* $\mathbf{h}$, *and* $SNR := d_{jk}/\sigma^2$. *Furthermore, if* $\mathbf{R}_h$ *has rank* r, *then ZF equalization with* $\mathbf{G}$ *provides* r^{th} *order diversity gain.*

Proof: We first need to upper bound the argument of the exponential in (8) with a multiple of $\parallel \mathbf{h} \parallel^2$. Towards this goal, we first express $\parallel \mathbf{G}^H \mathbf{e}_{jk} \parallel^2$ as

$$\parallel \mathbf{G}^H \mathbf{e}_{jk} \parallel^2 = \mathbf{e}_{jk}^H \mathbf{\Theta}_\kappa \mathbf{D}_\kappa \mathbf{D}_\kappa^H \mathbf{\Theta}_\kappa^H \mathbf{e}_{jk} \ . \tag{9}$$

Let us now define $\mathbf{c}_\kappa := \mathbf{\Theta}_\kappa^H \mathbf{e}_{jk}$, where in defining $\mathbf{c}_\kappa$, we have dropped the dependence on codeword indices j and k for convenience. Recalling the definition of $\mathbf{D}_\kappa$, we can now express (9) as

$$\parallel \mathbf{G}^H \mathbf{e}_{jk} \parallel^2 = \sum_{p=0}^{P-1} |c_\kappa(p)|^2 |H(e^{-j2\pi p/P})|^{-2} \ , \tag{10}$$

where $c_\kappa(p)$ is the p^{th} entry of $\mathbf{c}_\kappa$. Notice that $c_\kappa(p) = 0$ whenever $p \in \kappa$ because $\mathbf{\Theta}_\kappa^H$ has zero rows in indices belonging to κ. We would now like to find an upper bound to (10). Let $p_0 := \mathrm{argmax}_p |c_\kappa(p)|^2 |H(e^{-j2\pi p/P})|^{-2}$. Note that $p_0 \notin \kappa$ because $c_\kappa(p) = 0$ whenever $p \in \kappa$. Furthermore, it is straightforward that (10) is upper bounded by $N|c_\kappa(p_0)|^2 |H(e^{2\pi p_0/P})|^{-2}$. This means that the right hand side of (8) is upper bounded by

$$\frac{1}{2}\exp\left(\frac{-d_{jk}^2 |H(e^{2\pi p_0/P})|^2}{8\sigma^2 |c_\kappa(p_0)|^2 N}\right) \ . \tag{11}$$

Now, let

$$\mathbf{H}_\kappa := [H(e^{-j2\pi p_0/P}) H(e^{-j2\pi p_1/P}) \ldots H(e^{-j2\pi p_{\bar{L}}/P})]^T \ .$$

$\mathbf{H}_\kappa$ can be obtained from $\mathbf{h}$ through a linear transformation:

$$\mathbf{H}_\kappa = \mathbf{V}_\kappa \mathbf{h} \ , \qquad (12)$$

where $[\mathbf{V}_\kappa]_{m,l} = \exp(-jp_m l/P)$ is Vandermonde. Let $\lambda_{min}(\cdot)$ denote the minimum eigenvalue. Then using (12), and a standard result from matrix theory [4, pp.534] $\lambda_{min}(\mathbf{V}_\kappa^H \mathbf{V}_\kappa) \parallel \mathbf{h} \parallel^2 \leq \parallel \mathbf{H}_\kappa \parallel^2$ holds. This can be used to show that

$$\begin{aligned} \lambda_{min}(\mathbf{V}_\kappa^H \mathbf{V}_\kappa) \parallel \mathbf{h} \parallel^2 &\leq \sum_{l=0}^{\bar{L}} |H(e^{-j2\pi p_l/P})|^2 \\ &\leq (\bar{L}+1)|H(e^{-j2\pi p_0/P})|^2 \ , \end{aligned}$$

because $|H(e^{-j2\pi p_0/P})| \geq |H(e^{-j2\pi p_l/P})|, l = 1, \ldots, \bar{L}$. This leads us to conclude that

$$|H(e^{-j2\pi p_0/P})|^2 \geq \frac{1}{\bar{L}+1} \lambda_{min}(\mathbf{V}_\kappa^H \mathbf{V}_\kappa) \parallel \mathbf{h} \parallel^2 \ . \qquad (13)$$

We can finally use (13) along with (11) to upper bound (11) as

$$\frac{1}{2} \exp\left(\frac{-d_{jk}^2 \lambda_{min}(\mathbf{V}_\kappa^H \mathbf{V}_\kappa) \parallel \mathbf{h} \parallel^2}{8\sigma^2 |c_\kappa(p_0)|^2 N(\bar{L}+1)} \right) \ , \qquad (14)$$

which is also an upper bound to (8). We would like to average (14) over the channel statistics. The terms that depend on the instantaneous realization of the channel are $\mathbf{h}, \mathbf{V}_\kappa$, and $c_\kappa(p_0)$. But since $\mathbf{V}_\kappa$ is always full rank (recall that it is a Vandermonde matrix) $\lambda_{min}(\mathbf{V}_\kappa^H \mathbf{V}_\kappa) > 0$, regardless of the indices in κ. Similarly, $|c_\kappa(p_0)| < \infty$ for any set of indices κ because $\mathbf{\Theta}$ is Vandermonde, implying that $\mathbf{A}$ will always be full rank. We can then conclude that (14) can be further upper bounded by

$$\frac{1}{2} \exp\left(\frac{-d_{jk} \lambda_{min} \parallel \mathbf{h} \parallel^2}{8\sigma^2 CN(\bar{L}+1)} \right) \ , \qquad (15)$$

where $\lambda_{min} = \min(\lambda_{min}(\mathbf{V}_\kappa^H \mathbf{V}_\kappa))$, and $C = \max(|c_\kappa(p_0)|^2)$ with the minimization and maximization taken over κ ranging over all subsets of $\{0, \ldots, P-1\}$ with $\bar{L}$ elements. Hence we have proved the first part of the theorem with $K := \lambda_{min}/(8CN(\bar{L}+1))$, which is evidently independent of $\mathbf{h}$.

We are now ready to average (15) with respect to the distribution of $\mathbf{h}$, which is assumed to be zero-mean Gaussian with correlation matrix $\mathbf{R}_h$. Let $\lambda_l, l = 0, \ldots, L$ be the eigenvalues of $\mathbf{R}_h$. It is well-known that

$$E_{\mathbf{h}}\left[\exp\left(-K\text{SNR} \parallel \mathbf{h} \parallel^2\right)\right] \leq (K\ \text{SNR})^{-r} \left(\prod_{l=0}^{r-1} \lambda_l \right)^{-1} ,$$

where $E_{\mathbf{h}}[\cdot]$ means expectation with respect to the distribution of $\mathbf{h}$. Hence we have shown that the linear equalizer proposed in (7) provides full multipath diversity, given by the rank of the channel covariance matrix which concludes the proof. Notice that if the channel has iid taps, than the diversity orders is $r = (L+1)$.

3. EQUALIZER COMPLEXITY AND PERFORMANCE

The complexity of a brute force ML receiver for the I/O relationship in (1) is $\mathcal{O}(M^N)$ where M is the size of the constellation transmitted in each entry of the vector $\mathbf{s}[i]$. This is prohibitively complex for reasonable values of N, which motivates our linear equalization scheme as a viable alternative.

The computation of the equalizer output $\mathbf{Gx}$ entails calculating $\mathbf{\Theta}_\kappa \mathbf{D}_\kappa \mathbf{Fx}$ for each block index i. This means we need to compute the P point FFT of $\mathbf{x}$, to obtain $\mathbf{y} := \mathbf{Fx}$, which requires $P \log P$ operations. Computation of $\mathbf{D}_\kappa$ and multiplication by it entails sorting $|H(e^{-j2\pi p/P})|$ to identify κ which requires $2N$ real multiplications, which is equivalent to N complex multiplications (since the subtractions and comparisons involved in sorting $|H(e^{-j2\pi p/P})|$ is negligible, we ignore them in the computational count). Then the p^{th} element of vector $\mathbf{z}$ is calculated as $[\mathbf{z}]_p := [\mathbf{y}]_p / H(e^{-j2\pi p/P})$ if $p \notin \kappa$ and $[\mathbf{z}]_p = 0$ otherwise. This requires N division operations and gives us $\mathbf{z} = \mathbf{D}_\kappa \mathbf{Fx}$. We next need to solve the linear system of equation given by $(\mathbf{I}_\kappa \mathbf{\Theta})\hat{\mathbf{s}} = \mathbf{z}$, where $\mathbf{I}_\kappa$ is a diagonal matrix with zeros and ones along the diagonal with $[\mathbf{I}_\kappa]_{p,p} = 1$ if $p \notin \kappa$, and $[\mathbf{I}_\kappa]_{p,p} = 0$ otherwise. Since $\mathbf{\Theta}$ is a Vandermonde matrix, the solution of this equation can be found efficiently in $5N^2/2$ operations [1, pp. 185]. Hence the linear equalization of the data has a complexity given approximately by $5N/2 + (P/N)\log P + 2$ per symbol (which is obtained by adding the complexity associated with each stage and normalizing the result with the block length N). Please note that, for reasonable constellation sizes and channel lengths, this is much less than the complexity of the Viterbi algorithm (which is $\mathcal{O}(M^L)$), or the calculation of the pseudo-inverse of $\mathbf{H}$ (which is $\mathcal{O}(N^2)$ per symbol). Notice that the complexity count we have just performed for the proposed algorithm is the worst-case complexity, where the solution to the Vandermonde system of equations is performed for every block. If the channel is approximately the same across blocks, so that κ has not changed, the complexity can further be reduced, relying on the computations used in the previous block.

However, the equalizer $\mathbf{G}$ in (7), despite its maximum diversity at high SNRs, does not perform very well at low SNRs. The primary reason for this is that $\mathbf{A}_\kappa$ has a very large condition number when $\bar{L}$, the cardinality of κ, is large, resulting in a $\mathbf{\Theta}_\kappa$ with large elements. This results in an equalizer that greatly amplifies the noise, not because of the fading channel, but because $\mathbf{\Theta}_\kappa$ obtained from $\mathbf{A}_\kappa^{-1}$ has large entries. One approach to overcoming this problem is to redefine κ to have fewer than $\bar{L}$ elements so that $\mathbf{\Theta}_\kappa$ will not be ill conditioned. It can be shown that this will result in a reduction in the diversity gain, but the improvement of performance over the low-medium SNR range. We next illustrate these points in the simulations.

4. SIMULATIONS AND CONCLUSIONS

The main result of this paper is that an increase in number of multipath components will improve the performance over fading channels even when linear zero-forcing equalization is used. We see this in Figure 1, where the pseudoinverse equalizer is used, and the Frame Error Rate performance for $L = 0$ and $L = 3$ (a one-tap and a four-tap channel respectively) is shown. We observe the performance improvement when the average power of the channel $\parallel \mathbf{h} \parallel$ is fixed, but the number of taps is increased. Hence, the gains in performance with increase in multipath diversity is evident in Figure 1 when the pseudoinverse equalizer is utilized. That the pseudoinverse equalizer achieves MMD is shown in [5], in a similar context, by using the fact that (7) achieves MMD and that the pseudoinverse has the minimum norm property.

As mentioned in the previous section, the performance of $\mathbf{G}$ in (7) degrades rapidly as the cardinality of κ, $|\kappa|$ increases. This problem can be overcome by keeping $|\kappa|$ small. So instead of defining κ to be the indices corresponding to the $\bar{L}$ smallest channel frequency responses, we redefine κ to have the m smallest channel frequency responses, where $m < \bar{L}$. We show the frame error rates corresponding to these results in Figure 2. We observe that having $|\kappa| = 0$, which corresponds to using an equalizer given by $\mathbf{\Theta}^{\dagger}\mathbf{D}^{-1}\mathbf{F}$, provides no diversity for both $L = 0$, and $L = 3$. On the other hand, when $|\kappa|$ is increased to 1, we see that the performance is better due to the increased slope of the curve, when $L = 3$. When $|\kappa| = 2$ (not shown), the performance degrades due to the ill-conditioning of $\mathbf{A}_{\kappa}$, for the SNR range shown in the figure. We also observe that the equalizers in Figure 2 do substantially worse than the pseudoinverse equalizer in Figure 1. This motivates looking investigating the implementation of the pseudoinverse of $\mathbf{H}$ exploiting its banded Toeplitz structure using a similar approach to that used in $\mathbf{G}$.

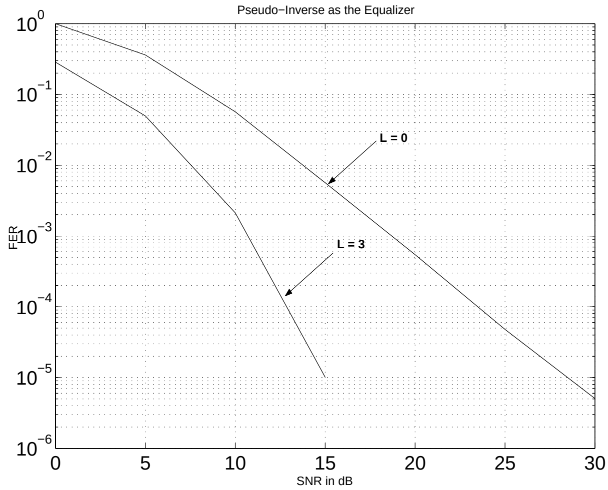

Fig. 1. Performance of the pseudo-inverse

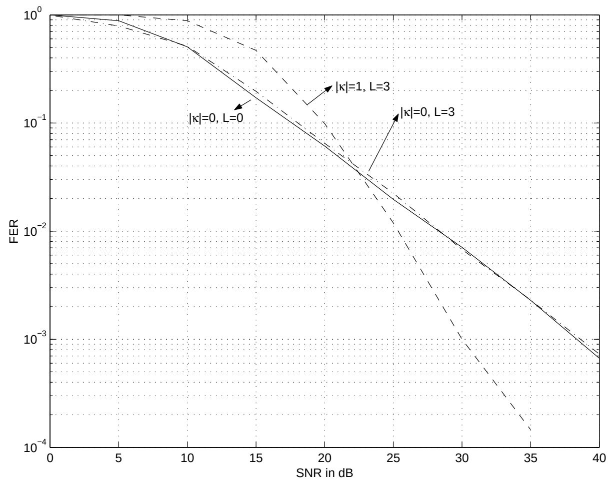

Fig. 2. Performance for different $|\kappa|$

5. REFERENCES

[1] G. H. Golub and C. F. Van Loan, *Matrix Computations*, John's Hopkins University Press, 1996.

[2] J. Proakis, *Digital Communications*, McGraw-Hill, 1995.

[3] M. K. Simon, S. M. Hinedi, W. C. Lindsey, *Digital Communication Techniques*, Prentice Hall, 1995.

[4] T. Söderström, P. Stoica, *System Identification*, Prentice Hall, 1989.

[5] C. Tepedelenlioglu, "Maximum Multipath Diversity with Linear Equalization for linearly precoded OFDM systems," ISIT 2003, (submitted).

[6] Z. Wang and G. B. Giannakis. "Wireless multicarrier communications: where Fourier meets Shannon," *IEEE Signal Proc. Mag.*, 47(3):29-48, May 2000.

[7] Z. Wang and G. B. Giannakis, "Linearly precoded or coded OFDM against wireless channel fades?" SPAWC '01, pp. 267-270.

[8] Z. Wang, X. Ma, and G. B. Giannakis. "Optimality of single-carrier zero-padded block transmissions". In *Proc. of WCNC*, pp. 660-664, Orlando, FL, March 17-21 2002.