ON THE DISCRETE COSINE TRANSFORM AND OFDM SYSTEMS

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ABSTRACT

A method for using the discrete cosine transform (DCT) as an alternative to the discrete Fourier transform (DFT) for orthogonal frequency division multiplexing (OFDM) wireless transmission methods is presented. These transforms satisfy the cyclic convolution properties of the DFT when used with a symmetric extension. Analysis of intersymbol and intercarrier interference in OFDM systems reveals that under certain channel conditions, throughput is enhanced when using the DCT rather than the DFT.

1. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) systems are attractive for wireless communications due to several different reasons, including the inherent frequency diversity due to multicarrier modulation. However, channel equalization in an OFDM system is a computationally expensive task, particularly for high-bandwidth OFDM systems where processing time at the receiver may be limited.

The means of doing away with the need for equalization in an OFDM system is based on the use of a *guard interval* and a *cyclic prefix* (which will be described later). The guard interval describes a transmission period between OFDM symbols. If the guard interval is larger than the maximum delay spread of the channel, then intersymbol interference (ISI) does not become problematic. However, in frequency selective channels, the problem of intercarrier interference (ICI) is not simply solved by the use of a guard interval.

The basic OFDM transmission system is given in Figure 1. With reference to this figure, $\{a_i(0), a_i(1), \ldots, a_i(N-1)\}$ comprises the input data vector for OFDM symbol *i*, $\{x_i(0), x_i(1), \ldots, x_i(N-1)\}$ the actual OFDM symbol to be transmitted, *x*(*k*) the information transmitted after addition of the guard interval, *r*(*k*) the received data, $\{r_i(0), r_i(1), \ldots, r_i(N-1)\}$ the received vector after elimination of the guard interval, and $\{R_i(0), R_i(1), \ldots, R_i(N-1)\}$ the demodulated information vector.

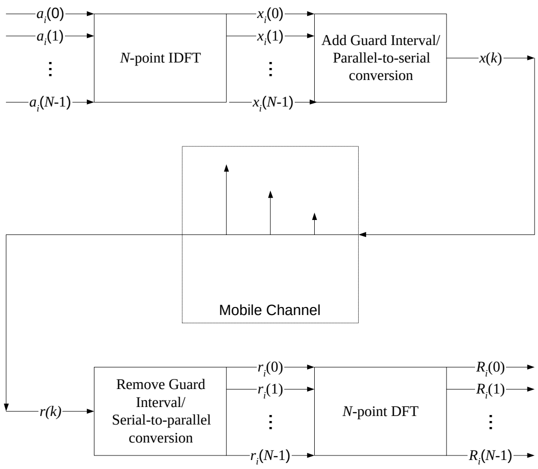

Figure 1: OFDM Transmission

Since OFDM systems normally use the discrete Fourier transform (DFT) for multicarrier modulation of the data to be transmitted, the user information after applying the DFT transformation follows the cyclic shift properties of the DFT matrix (namely the multiplication of its transform coefficients by a complex exponential whose argument is linearly related to the temporal shift). By applying a *cyclic prefix*, merely involves pre-pending some number of the ending data vector entries of $\{x_i(0), x_i(1), \ldots, x_i(N-1)\}$ to the beginning of the OFDM symbol to be transmitted, then the interference per subcarrier will resemble a flat fading channel, as long as the maximum delay spread of the channel is less than the length of the cyclic prefix.

Cyclic shift properties are not unique to the DFT. In fact, cyclic shift properties were extended to a wide variety of sinusoidal transforms in Martucci's work [1], where the author showed how cyclic shift properties can be derived for the discrete cosine transform (DCT) through the use of *symmetric extension*. Symmetric extension involves the replicating of a sequence such that the resultant sequence is either symmetric or asymmetric. The consequence of such an extension is the reduction in

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throughput by a factor of at least one-half when compared to the DFT matrix. Therefore, these types of transforms (when used with a cyclic prefix) should be considered as an alternative to the DFT only for wireless channel profiles where a potential gain in overall throughput (taking into account the throughput-reducing effects of the symmetric extension) justifies their use.

2. CYCLIC SHIFT PROPERTIES OF THE DCT

The DCT does do not have a cyclic shift (and therefore cyclic convolution) property in and of itself. However, cyclic shift properties arise when a symmetric extension is applied. We can see that a symmetric extension of the input vector to a DCT operation will result in half of the subcarriers being equivalent to zero. This can be seen starting with the definition of the DCT:

$$C[m]=\sum_{l=0}^{N-1}\sqrt{\frac{k_m}{N}}\cos\left(\frac{\pi(2l+1)m}{2N}\right)x[l],\qquad k_m=\begin{cases}1, & m=0\\ 2, & m\neq 0\end{cases} \tag{1}$$

In (1), the N-length input vector is $x[l]$, the N-length output vector is $C[m]$, and k_m is a scaling constant that is dependent on the subcarrier index m. When the input vector is symmetric, i.e.,

$$x[l]=\left\{x[0],\ldots,x\left[\frac{N}{2}-1\right],x\left[\frac{N}{2}-1\right],\ldots,x[0]\right\} \tag{2}$$

then the output DCT coefficients become

$$C[m]=\sum_{l=0}^{\frac{N}{2}-1}\sqrt{\frac{k_m}{N}}x[l]\left(\cos\left(\frac{\pi(2l+1)m}{2N}\right)+\cos\left(\frac{\pi(2N-2l-1)m}{2N}\right)\right) \tag{3}$$

The sum in (3) becomes 0 when m is odd. This is determined by breaking down the cosine terms:

$$\begin{aligned}&\cos\left(\frac{\pi(2l+1)m}{2N}\right)+\cos\left(\frac{\pi(2N-2l-1)m}{2N}\right)=\\&\cos\left(\frac{\pi 2lm}{2N}\right)\cos\left(\frac{\pi m}{2N}\right)-\sin\left(\frac{\pi 2lm}{2N}\right)\sin\left(\frac{\pi m}{2N}\right)+\\&\cos\left(\frac{\pi(2N-2l)m}{2N}\right)\cos\left(\frac{\pi m}{2N}\right)+\\&\sin\left(\frac{\pi(2N-2l)m}{2N}\right)\sin\left(\frac{\pi m}{2N}\right)\end{aligned} \tag{4}$$

This term can be further expanded:

$$\begin{aligned}&\cos\left(\frac{\pi 2lm}{2N}\right)\cos\left(\frac{\pi m}{2N}\right)-\sin\left(\frac{\pi 2lm}{2N}\right)\sin\left(\frac{\pi m}{2N}\right)+\\&\cos(\pi m)\cos\left(\frac{\pi 2lm}{2N}\right)\cos\left(\frac{\pi m}{2N}\right)+\sin(\pi m)\sin\left(\frac{\pi 2lm}{2N}\right)\cos\left(\frac{\pi m}{2N}\right)+\\&\sin(\pi m)\cos\left(\frac{\pi 2lm}{2N}\right)\sin\left(\frac{\pi m}{2N}\right)-\cos(\pi m)\sin\left(\frac{\pi 2lm}{2N}\right)\sin\left(\frac{\pi m}{2N}\right)\end{aligned} \tag{5}$$

If m is odd, then (5) becomes

$$\begin{aligned}&\cos\left(\frac{\pi 2lm}{2N}\right)\cos\left(\frac{\pi m}{2N}\right)-\sin\left(\frac{\pi 2lm}{2N}\right)\sin\left(\frac{\pi m}{2N}\right)+\\&-\cos\left(\frac{\pi 2lm}{2N}\right)\cos\left(\frac{\pi m}{2N}\right)+\sin\left(\frac{\pi 2lm}{2N}\right)\sin\left(\frac{\pi m}{2N}\right)=0\end{aligned} \tag{6}$$

The results of a cyclic shift therefore must be analyzed on a subcarrier-by-subcarrier basis. Assuming a cyclic shift by an integer quantity r, the resultant DCT coefficients may be expressed as

$$C_r[m]=\sum_{l=0}^{\frac{N}{2}-1}\sqrt{\frac{k_m}{N}}x[((l+r))_N]\left(\cos\left(\frac{\pi(2l+1)m}{2N}\right)+\cos\left(\frac{\pi(2(N-l-1)+1)m}{2N}\right)\right) \tag{7}$$

With respect to (3) and (7), $C_r[m] = C[m]$ when $r = 0$. In (7), it can be seen that the DC-coefficient ($m = 0$) will always be the same regardless of cyclic shift. For cyclic shifts greater than zero for *even* coefficients, the following applies:

$$\begin{aligned}C_r[m]&=\sum_{l=0}^{\frac{N}{2}-1}\sqrt{\frac{k_m}{N}}x[((l+r))_N]\left(\cos\left(\frac{\pi(2l+1)m}{2N}\right)+\cos\left(\frac{\pi(2(N-l-1)+1)m}{2N}\right)\right)\\&=\sum_{l=0}^{\frac{N}{2}-1}\sqrt{\frac{k_m}{N}}x[l]\left(\cos\left(\frac{\pi(2((l+r))_N+1)m}{2N}\right)+\cos\left(\frac{\pi(2((N-l-1+r))_N+1)m}{2N}\right)\right)\\&=\cos\left(\frac{\pi rm}{N}\right)C[m]\end{aligned} \tag{8}$$

3. USE OF THE DCT IN AN OFDM SYSTEM

Sinusoidal transforms in an OFDM system, the basic transmission can also be determined. Clearly, the throughput of such a system (without considering mobile channel effects) is reduced by a factor of two with respect to the comparable DFT-based method of transmission, based on the inclusion of an extension. The transmission system is depicted in **Figure 2**.

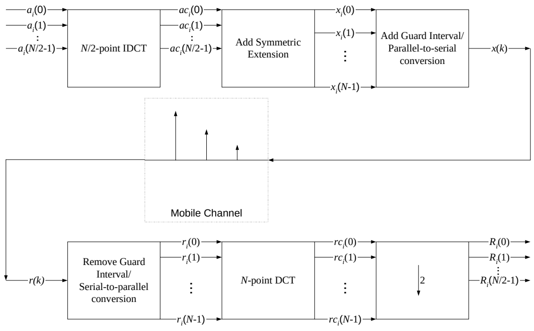

Figure 2: OFDM Transmission with Sinusoidal Transforms

With respect to Figure 2, in the case of the IDCT the output of the *N*/2-point inverse transform operation (including a scaling by a factor of the square root of 2) is

$$ac_i^{g}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N/2-1} a_i(n)\sqrt{k_n}\cos\left(\frac{\pi(2k+1)n}{N}\right), \quad 0 \le k \le N/2-1 \tag{9}$$

The actual transmitted symbol (excluding the guard interval) is

$$\begin{aligned} x_i^{g}(k) &= ac_i(k),\ 0 \le k \le N/2-1 \\ x_i^{g}(k) &= \pm ac_i(N-1-k),\ N/2 \le k \le N-1 \end{aligned} \tag{10}$$

An *N*-point DCT can be used to recover the original *N*/2 information symbols $a_i(n)$. This can be seen when the transform is applied to the transmitted information symbols after elimination of the cyclic extension, the following results:

$$R_i(l) = \frac{1}{N}\sum_{n=0}^{N/2-1}\sqrt{k_l k_n}\,a_i(n)\times \sum_{k=0}^{N/2-1}\left[\begin{aligned} &\cos\left(\frac{\pi(2k+1)l}{2N}\right)\cos\left(\frac{\pi(2k+1)n}{N}\right)+ \\ &\cos\left(\frac{\pi(2k+1+N)l}{2N}\right)\cos\left(\frac{\pi(N-2k-1)n}{N}\right)\end{aligned}\right] \tag{11}$$

Since the input sequence to the transform is symmetric, only the even coefficients are significant (i.e. when l is even). Under these conditions, (11) becomes

$$R_i(l) = \frac{1}{N}\sum_{n=0}^{N/2-1}\sqrt{k_l k_n}\,a_i(n)\times \sum_{k=0}^{N/2-1}\left[\begin{aligned} &\cos\left(\frac{\pi(2k+1)l}{2N}\right)\cos\left(\frac{\pi(2k+1)n}{N}\right)+ \\ &\cos\left(\frac{\pi l}{2}\right)\cos\left(\frac{\pi(2k+1)l}{2N}\right)\cos(\pi n)\cos\left(\frac{\pi(2k+1)n}{N}\right)\end{aligned}\right] \tag{12}$$

Since l is even, two cases in (12) are significant, when $l = 2n$ and when $l = 2n + 1$. In the former case, the following occurs:

$$\begin{aligned} &R_i(l) = \frac{1}{N}\sum_{n=0}^{N/2-1}\sqrt{k_l k_n}\,a_i(n)\times \\ &\sum_{k=0}^{N/2-1}\left[\begin{aligned} &\cos\left(\frac{\pi(2k+1)2n}{2N}\right)\cos\left(\frac{\pi(2k+1)n}{N}\right)+ \\ &\cos\left(\frac{\pi 2n}{2}\right)\cos\left(\frac{\pi(2k+1)2n}{2N}\right)\cos(\pi n)\cos\left(\frac{\pi(2k+1)n}{N}\right)\end{aligned}\right] = \\ &\frac{1}{N}\sum_{n=0}^{N/2-1}\sqrt{k_l k_n}\,a_i(n)\sum_{k=0}^{N/2-1} 2\cos^2\left(\frac{\pi(2k+1)n}{N}\right) = 1 \end{aligned} \tag{13}$$

In the latter case,

$$\begin{aligned} &R_i(l) = \frac{1}{N}\sum_{n=0}^{N/2-1}\sqrt{k_l k_n}\,a_i(n)\times \\ &\sum_{k=0}^{N/2-1}\left[\begin{aligned} &\cos\left(\frac{\pi(2k+1)l}{2N}\right)\cos\left(\frac{\pi(2k+1)n}{N}\right)+ \\ &\cos\left(\frac{\pi l}{2}\right)\cos\left(\frac{\pi(2k+1)n}{N}\right)\cos(\pi n)\cos\left(\frac{\pi(2k+1)n}{N}\right)\end{aligned}\right] = \\ &\frac{1}{N}\sum_{n=0}^{N/2-1}\sqrt{k_l k_n}\,a_i(n)\times \\ &\sum_{k=0}^{N/2-1}\left[\begin{aligned} &\cos\left(\frac{\pi(2k+1)l}{2N}\right)\cos\left(\frac{\pi(2k+1)n}{N}\right)- \\ &\cos\left(\frac{\pi(2k+1)n}{N}\right)\cos\left(\frac{\pi(2k+1)n}{N}\right)\end{aligned}\right] = 0 \end{aligned} \tag{14}$$

Therefore the *N*-by-*N* DCT matrix is orthogonal to the *N*/2-by-*N* IDCT matrix representing the transmitter inverse transformation operation followed by the symmetric extension and therefore recovers the original transmitted symbol sequence $a_i(n)$.

4. INTERSYMBOL AND INTERCARRIER INTERFERENCE

In [2] and [3], different approaches to analysis of the performance of DFT-based OFDM systems is provided with respect to wireless channels which introduce intersymbol interference (ISI). DFT-based OFDM systems are able to simply handle wireless channels whose delay spread is shorter than the guard interval, but suffer when channel conditions result in significant multipaths falling outside of the guard interval.

While transforms such as the DCT result in a loss of throughput due to the symmetric extension when used in OFDM, these transforms may outperform the DFT in ISI channels. The effects of ISI and intercarrier interference (ICI) resulting from wireless channels whose delay spread is greater than the cyclic prefix can be examined using the analysis of [3]. With respect to Figure 1, the output vector for the IDFT operation at OFDM symbol index i is (when the guard interval is G subcarriers) is

$$x_i^{g}(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} a_i(n)e^{j\frac{2\pi nk}{N}},\ -G \le k \le N-1 \tag{15}$$

The first G elements of $x_i^g(k)$ $-G \leq k \leq -1$, form the cyclic prefix. For an equivalent IDCT operation (see Figure 2), the output vectors are

$$x_i^g(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N/2-1} a_i(n)\sqrt{k_n} \cos\left(\frac{\pi(2k+1)n}{N}\right), \quad 0 \leq k \leq N/2-1 \tag{16}$$

$$x_i^g(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N/2-1} \left[a_i(n)\sqrt{k_n} \cos\left(\frac{\pi(2(N-k-1)+1)n}{N}\right)\right], \quad N/2 \leq k \leq N-1$$

$$x_i^g(k) = x_i^g(N+k), -G \leq k < 0$$

In the case of the DCT, the last G elements are used to form a cyclic extension so that it is length $N + G$. Neglecting any additive noise in the wireless channel, if the channel is of delay spread equivalent to M OFDM samples and has a *static* channel impulse response represented by the h_m, then the received sequence may be expressed as

$$r(k) = \sum_{i=-\infty}^{\infty} \sum_{m=0}^{M-1} h_m x_i^g(k-m-i(N+G)) \tag{17}$$

Assuming perfect synchronization at the receiver, the received sequence may be rearranged with respect to each OFDM symbol, i.e.

$$r_i(k) = r(i(N+G)+k), -G \leq k < N \tag{18}$$

Assuming that the channel is an ISI-channel, i.e. $M > G$, the received sequence may be represented as

$$r_i(k) = \sum_{m=0}^{k+G} h_m x_i^g(k-m) + \sum_{m=k+G+1}^{M-1} h_m x_{i-1}^g(N+G+k-m), \quad 0 \leq k \leq M-G-1 \tag{19}$$

$$= \sum_{m=0}^{M-1} h_m x_i^g(k-m), \; M-G \leq k < N-1$$

This received sequence of samples is then passed into either a DFT or DCT. The desired signal at subcarrier l (weighted by $H_i(l)$) after transformation, the ICI term $C_i(l)$, and ISI term $S_i(l)$ may be isolated:

$$R_i(l) = H_i(l)a_i(l) + C_i(l) + S_i(l) \tag{20}$$

The derivations of desired signal, ICI and ISI terms are not included here, but can be derived based on the definitions of the DCT and DFT

The analysis proposed in [2] attempted to characterize throughput assuming a fixed modulation per subcarrier. This kind of modulation does not maximize throughput, and the analysis presented in this section demonstrates the fluctuations in SNR for each subcarrier as a result of ISI channels. Therefore, it is desirable to find the modulation constellation that maximizes throughput on a *per subcarrier* basis [4]. The theoretical error bounds for a K-point rectangular QAM constellation (i.e. K is a power of 2) on a given subcarrier with SNR $\gamma(l)$ is given as:

$$P_K(\gamma(l), K) \leq 1 - \left[1 - 2Q\left(\sqrt{\frac{3\gamma(l)}{(K-1)}}\right)\right] \tag{21}$$

Assuming equivalency in (21), the number of bits per modulation constellation symbol may be assigned per subcarrier based on the following rule:

$$K(l) = \log_2 \left\lfloor \max_R R(1 - P_K(\gamma(l), R)) \right\rfloor \tag{22}$$

As a simple example of differences in throughput, the following channel was examined for $N = 64$ and $G = 4$:

$$h(m) = e^{-p(m+1)}, \; 0 \leq m < M \tag{23}$$

The throughput bound for the DCT-based method was 74.94 bits/symbol, while for the DFT-based method was 64.01 bits/symbol.

5. CONCLUSIONS

A method for OFDM transmission using the DCT and taking advantage of its cyclic convolutions properties was presented in this paper. In order to ensure cyclic convolution is possible, a symmetric extension of the input sequence to the DCT is necessary. However, in certain ISI channels, the DCT still provides throughput over the DFT.

11. REFERENCES

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