COMPUTATION OF THE CONTINUOUS-TIME PAR OF AN OFDM SIGNAL

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ABSTRACT

Computation of the peak-to-average envelope power ratio (PAR) of an orthogonal frequency division multiplexing (OFDM) signal plays a major role in peak reduction methods. A procedure for computing the continuous-time PAR of an OFDM signal, with any PSK or QAM constellations, is developed here. It is shown that the derivative of the instantaneous envelope power function (EPF) can be transformed into a linear sum of Chebyshev polynomials of the first kind and the second kind. Consequently, the roots of the derivative of EPF can be obtained by solving a polynomial. The procedure may be useful for theoretical studies of PAR distributions and developing the peak reduction methods.

1. INTRODUCTION

OFDM has become a popular technique in various high-speed wireless systems owing to the high spectrum efficiency and channel robustness [1]. The principal drawback of OFDM is that the PAR of an OFDM signal with N subcarriers may be up to N times that of a corresponding single carrier signal. A high PAR is undesirable in practice. In the digital part such as D/A and A/D convertors, it requires a large word length in order to keep the precision and quantization noise at an acceptable level. Consequently, they are used very inefficiently, as most of the signal amplitudes are only a fraction of the peak amplitude [1,2]. In the analog part, when the signal is applied to a non-linear device such as a RF power amplifier, it results in in-band distortion, which degrades the bit error rate performance, and out-of-band radiation, which reduces the spectral efficiency [1].

Two leading proposed approaches to resolving peak power problem are to improve the performance of power amplifier and to generate signals with low PAR. The amplifier, which can handle the peak power problem, needs to be highly linear or operated with a large back off. Both approaches result in a severe power efficiency penalty and are expensive [3]. This may have a deleterious effect on battery lifetime in mobile applications. Although many creative methods for reducing PAR have been proposed, almost all of them are devoted to peak power reduction of the discrete time signal [3-5]. In fact, when considering the analog signal processing, reducing the peak factor of a continuous wave is required. The PAR of the sampled sequence should not be larger than that of the continuous waveform, and reducing the former does not necessarily imply a similar reduction of the latter [6], even some reduction methods perform worse on the latter compared to the former [4]. In many cases, the continuous-time PAR is approximated using the discrete-time PAR [7], which is obtained by the Nyquist sampling or oversampling of the OFDM signal with IFFT, as obtaining the exact PAR value of continuous-time OFDM signal is difficult. Recently, a procedure using Chebyshev polynomials of the first kind solves the computation problem of the continuous-time PAR of an OFDM signal with BPSK subcarriers [8]. In this paper, we try to extend it to any PSK or QAM constellations.

To compute the continuous-time PAR, the roots of the derivative of the envelope power function (EPF) are required. But finding the required roots appears very difficult because the derivative of EPF is a sum of sinusoidal functions. As such, it may require the use of a general root finding algorithm for nonlinear functions, which is not easy to be implemented. In this paper, using an inverse cosine-based transformation, the derivative of the EPF can be converted to a sum of Chebyshev polynomials of the first kind and the second kind, and the required roots are now trapped within the interval -1 to 1. So the original root finding problem is reduced to a root finding problem for a polynomial. Reliable algorithms for finding all roots of a polynomial are well known [9]. Consequently, using this approach, the absolute peak of the envelope power function can be evaluated exactly.

2. MATHEMATICAL DEFINITIONS

The complex envelope of the OFDM signal with N subcarriers is represented by

$$s(t) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} S_n e^{j2\pi n \Delta f t}, \quad 0 \le t \le T, \qquad (1)$$

where $j=\sqrt{-1}$, $S_n = a_n + jb_n$ is complex symbols from a given PSK or QAM constellation, Δf is the subcarrier spacing and T is the symbol period. Since the cyclic prefix cannot introduce any new peaks in the symbol, we assume that $\Delta f = 1/T$. Also, for simplicity, we normalize time by T and then substitute $\theta = 2\pi t/T$ to get,

$$s(\theta)=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}S_n e^{jn\theta}, \qquad 0\le\theta\le 2\pi . \qquad (2)$$

The level of amplitude fluctuation of OFDM signals is measured in terms of peak factors that indicate the ratio of the peak power to the average power of the signal envelope. The continuous-time PAR for a given N-tuple of complex symbols $\vec{S}=(S_0,S_1,\cdots,S_{N-1})$ is defined as

$$PAR=\frac{\max\limits_{0\le\theta\le2\pi}|s(\theta)|^2}{P_{av}}, \qquad (3)$$

where P_{av}, a constant that depends on the signal constellation and N for an uncoded system, is the average envelope power of the signal. Most of PAR-reduction techniques are concerned with reducing this quantity of the transmitted signals to the level lower than certain specified threshold.

3. COMPUTATION OF CONTINUOUS-TIME PAR

To compute Eq.(3) exactly, the roots of $d|s(\theta)|^2/d\theta$ are needed. The instantaneous envelope power of the signal, i.e. $|s(\theta)|^2$, is a real-valued function and may be represented as

$$|s(\theta)|^2=\frac{1}{N}\sum_{n=0}^{N-1}\left(a_n^2+b_n^2\right)+\frac{2}{N}\sum_{k=1}^{N-1}\left(\alpha_k\cos(k\theta)+\beta_k\sin(k\theta)\right), 0\le\theta\le2\pi, \quad (4)$$

where

$$\alpha_k=\sum_{n=0}^{N-1-k}\left(a_n a_{n+k}+b_n b_{n+k}\right), \quad k=1,2,\cdots,N-1,$$

$$\beta_k=\sum_{n=0}^{N-1-k}\left(a_{n+k}b_n-a_n b_{n+k}\right), \quad k=1,2,\cdots,N-1.$$

Let us define

$$P(\theta)=\frac{N}{2}\cdot\frac{d|s(\theta)|^2}{d\theta}=\sum_{k=1}^{N-1}\left(k\beta_k\cos(k\theta)-k\alpha_k\sin(k\theta)\right), 0\le\theta\le2\pi. \quad (5)$$

At first, we consider the solution of $P(\theta)=0$ in the interval of $[0,\pi]$. Let $\theta=\cos^{-1}(x)$, where $-1\le x\le 1$. Substituting it in the Eq.(5) and noticing $\sin(\cos^{-1}x)=\sqrt{1-x^2}$, we get

$$Q_1(x)=P(\cos^{-1}x)=\sum_{k=1}^{N-1}\left(k\beta_k T_k(x)-k\alpha_k\sqrt{1-x^2}U_{k-1}(x)\right), -1\le x\le 1, \quad (6)$$

where

$$T_k(x)=\cos(k\cos^{-1}x),$$

$$U_k(x)=\frac{\sin\left((k+1)\cos^{-1}x\right)}{\sin(\cos^{-1}x)}.$$

$T_k(x)$ and $U_k(x)$ are the kth-order Chebyshev polynomial of the first kind and the second kind [10], respectively. The explicit expressions of $T_k(x)$ and $U_k(x)$ for any nonnegative integer k are available, for example, $T_0(x)=1$, $T_1(x)=x$, $T_2(x)=2x^2-1$, $U_0(x)=1$, $U_1(x)=2x$, $U_2(x)=4x^2-1$ and so on.

Let $Q_1(x)=0$, we get

$$\sum_{k=1}^{N-1}k\beta_k T_k(x)=\sqrt{1-x^2}\sum_{k=1}^{N-1}k\alpha_k U_{k-1}(x), -1\le x\le 1 . \quad (7)$$

Squaring the two side of Eq.(7) and shifting the right term to the left, we will have

$$Q(x)=\left(\sum_{k=1}^{N-1}k\beta_k T_k(x)\right)^2-(1-x^2)\left(\sum_{k=1}^{N-1}k\alpha_k U_{k-1}(x)\right)^2, \quad -1\le x\le 1. \quad (8)$$

Obviously, the roots of $Q_1(x)$ must be the roots of $Q(x)$. Now, we consider the solution of $P(\theta)=0$ in the interval of $[\pi,2\pi]$. Let $\theta = 2\pi-\cos^{-1}(x)$, where $-1\le x\le 1$. Substituting it in the Eq.(5) and using the trigonometric identical equation, we then get

$$Q_2(x)=P(2\pi-\cos^{-1}x)=\sum_{k=1}^{N-1}\left(k\beta_k T_k(x)+k\alpha_k\sqrt{1-x^2}U_{k-1}(x)\right), -1\le x\le 1. \quad (9)$$

Just as we can get $Q(x)$ from $Q_1(x)$, we can get $Q(x)$ from $Q_2(x)$. So, all the roots of $Q_2(x)$ are also the roots of $Q(x)$.

Since the square of a polynomial is also a polynomial, $Q(x)$ is a polynomial of degree $(2N-2)$, with $(2N-2)$ roots, real or complex. Substituting the real roots of $Q(x)$ that lie between -1 and $+1$ into $Q_1(x)$ and $Q_2(x)$ and testing whether the result is zero or not, we can get the real roots of $Q_1(x)$ and $Q_2(x)$. Let the real roots of $Q_1(x)$ be $x_1^{(1)},x_2^{(1)},\cdots,x_{M_1}^{(1)}$, where $M_1\le 2N-2$, and the real roots of $Q_2(x)$ be $x_1^{(2)},x_2^{(2)},\cdots,x_{M_2}^{(2)}$, where $M_2\le 2N-2$. Define the set

$$\Lambda=\left\{\cos^{-1}\left(x_1^{(1)}\right),\cdots,\cos^{-1}\left(x_{M_1}^{(1)}\right),2\pi-\cos^{-1}\left(x_1^{(2)}\right),\cdots,2\pi-\cos^{-1}\left(x_{M_2}^{(2)}\right)\right\}. \quad (10)$$

It is clear that all the required roots of $P(\theta)$ are in this set. Therefore, according to the maximum value theory of

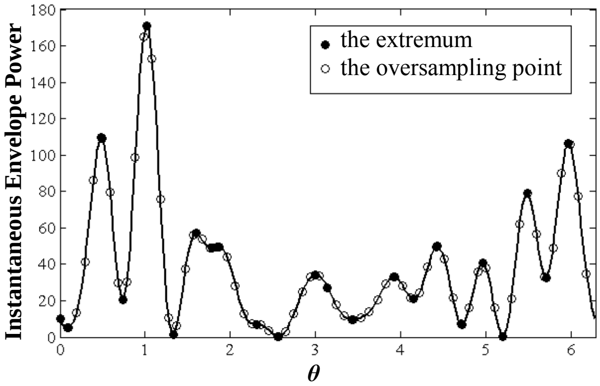

Fig.1. The extrema on the set Θ and the oversampling (L=4) points of $|s(\theta)|^2$ for a 16-carrier 64-QAM OFDM signal.

continuous function on an interval, the continuous-time PAR is obtained by

$$PAR = \frac{\max_{\theta \in \Theta} |s(\theta)|^2}{P_{av}}, \qquad (11)$$

where $\Theta = \Lambda \cup \{0, \pi\}$, is the extremum point set of instantaneous envelope power function, $|s(\theta)|^2$.

The following is an outline of the complete procedure.

Step 1) Calculate the coefficients α_k and β_k for a given N-tuple $\vec{S} = (S_0, S_1, \cdots, S_{N-1})$.

Step 2) Obtain the coefficients of $Q(x)$ in Eq.(8).

Step 3) Solve for the roots of the polynomial $Q(x)$.

Step 4) Find out the real roots of $Q_1(x)$ and $Q_2(x)$ and then obtain the real roots set Λ of $P(\theta)$.

Step 5) Evaluate $|s(\theta)|^2$ on Θ and pick the maximum.

4. CONCLUSIONS

From Fig. 1, which shows the results using the above method for a 16-carrier 64-QAM OFDM signal, we can see that all the extrema of the instantaneous envelope power function can be found and the maximum value point does not just lie in an oversampling point. So, the exact PAR of a continuous-time OFDM signal can definitely be found out by using this method, but it may not necessarily be found out by using oversampling.

Computation of the peak power plays a major role in peak reduction methods. Although Tellambura [8] has introduced a procedure for computing the continuous-time PAR of an OFDM signal, it is noneffective for other signal constellations except BPSK. In addition, Sharif et al. [7] have introduced an error bound for using discrete-time PAR to approximate the continuous-time PAR. But for the oversampling factor of four, which is usually considered accurate enough [4,7,8], the error bound theorem shows that the ratio of the continuous-time PAR to the discrete-time PAR may reach 2.09, which is too large for practical application. In this paper, a more accurate computational method for the continuous-time PAR of an OFDM signal, with any PSK or QAM constellations, has been developed. It may be useful for theoretical studies of PAR distributions and developing the peak reduction methods.

5. REFERENCES

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