OPTIMAL POWER ALLOCATION FOR MISO SYSTEMS AND COMPLETE CHARACTERIZATION OF THE IMPACT OF CORRELATION ON THE CAPACITY

H. Boche

Technical University of Berlin
Electrical Engineering and Computer Science
Heinrich-Hertz-Chair for
Mobile Communication Technology
Einsteinufer 25, D-10587 Berlin, Germany

E. A. Jorswieck *

Heinrich-Hertz-Institut für Nachrichtentechnik
Einsteinufer 37,
D-10587 Berlin, Germany,
jorswieck@hhi.de

ABSTRACT

We study the optimal transmission strategy of a multiple-input single-output (MISO) wireless communication link. The receiver has perfect channel state information while the transmitter has only long-term channel state information in regards of the channel covariance matrix. It was recently shown that the optimal eigenvectors of the transmit covariance matrix correspond with the eigenvalues of the channel covariance matrix. However, the optimal eigenvalues are difficult to compute. We develop a new characterization of the optimum power allocation. Furthermore, we apply this result to develop a simple algorithm which computes the optimum power allocation. In addition to this, we study the impact of correlation on the ergodic capacity of the MISO system with different channel state information (CSI) schemes. We show that the ergodic capacity with perfect CSI and without CSI at the transmitter is Schur-concave. Additionally, we show that the ergodic capacity with covariance knowledge at the transmitter is Schur-convex with respect to the correlation properties. Finally, we illustrate all theoretical results by numerical simulations.

1. INTRODUCTION

It is well known [1] that multiple-element antenna arrays can improve the performance and capacity of a wireless communication system in a fading environment. Especially multiple antennas at the transmitter have been studied frequently in the downlink beamforming scenario [2, 3], for example.

We consider the multiple-input single-output (MISO) single user case with imperfect channel state information at the transmit array. It was shown that even partial channel state information at the transmitter (CSIT) can increase the capacity of a MISO system. Recently, transmission schemes for optimizing capacity in MISO mean-feedback and covariance-feedback systems were derived in [4, 5]. The capacity can be achieved by Gaussian distributed transmit signals with a particular covariance matrix. Additionally, it was proven that the optimal transmit covariance matrix has the same eigenvectors as the known channel covariance matrix.

In this work, we characterize the optimum power allocation by a new necessary and sufficient condition. As a result, we develop a computational efficient algorithm which computes the optimum power allocation. Furthermore, we study the impact of correlation on the ergodic capacity of the MISO system with different CSI schemes. We show that the ergodic capacity is Schur-concave with perfect and without CSI at the transmitter while the ergodic capacity is Schur-convex in the case of covariance feedback. Furthermore, we illustrate all theoretical results by numerical simulations.

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2. SIGNAL MODEL

We consider the common MISO transmission model. We split the input data stream with identically independent distributed symbols with variance one $d(k)$ into m parallel data streams $d_1(k)$ $d_2(k), ..., d_m(k)$. Each parallel data stream is multiplied by a factor $\sqrt{\lambda_1}, ..., \sqrt{\lambda_m}$ and then weighted by a beamforming vector $\mathbf{u}_1, ..., \mathbf{u}_m$, respectively. The number of parallel data streams is less than or equal to the number of transmit antennas ($m \leq n_T$). The beamforming vectors have the size $1 \times n_T$ with n_T as the number of transmit antennas. The n_T signals of each weighted data stream $\mathbf{x}^i(k) = d_i(k) \cdot \sqrt{\lambda_i} \cdot \mathbf{u}_i$ are added up $\mathbf{x}(k) = \sum_{i=1}^{m} \mathbf{x}^i(k)$ and sent. By omitting the time index k for convenience we obtain in front of the transmit antennas

$$\mathbf{x} = \sum_{l=1}^{m} d_l \cdot \sqrt{\lambda_l} \cdot \mathbf{u}_l. \tag{1}$$

In matrix form we obtain for $\mathbf{x}$ with $\mathbf{\Lambda}_t = [\sqrt{\lambda}_1, ..., \sqrt{\lambda}_m]$ and $\mathbf{U} = [\mathbf{u}_1^T, ..., \mathbf{u}_m^T]^T$

$$\mathbf{x}^T = \mathbf{\Lambda}_t \cdot \mathbf{U}.$$

The transmit covariance matrix $\mathbf{Q} = \mathcal{E}(\mathbf{x}\mathbf{x}^H)$ is with (1) given by

$$\mathbf{Q} = \mathcal{E}\left(\mathbf{U}\mathbf{D}_\Lambda\mathbf{U}^H\right) \tag{2}$$

with diagonal matrix $\mathbf{D}_\Lambda = diag(\lambda_1, ..., \lambda_{n_T})$ with the eigenvalues. The flat fading channel model is given by

$$y = \mathbf{x}^H\mathbf{h} + n \tag{3}$$

with complex $n_T \times 1$ transmit vector $\mathbf{x}$, channel vector $\mathbf{h} = [h_1, ..., h_{n_T}]^T$ and circularly symmetric complex Gaussian noise n with variance $\frac{\sigma_n^2}{2}$ per dimension. $[\cdot]^T$ denotes transpose of a matrix of a vector and $[\cdot]^H$ denotes conjugate transpose of a matrix or a vector. It is assumed that the receiver perfectly knows $\mathbf{h}$. The transmitter receives feedback information from the receiver regarding $\mathbf{h}$ and knows that the channel is complex Gaussian distributed $\mathcal{CN}(\mathbf{0}, \Sigma)$ with covariance matrix Σ [4].

It was shown [4] that the capacity of such a system is given by

$$C^{cfCSI} = \max_{\mathbf{Q}:tr(\mathbf{Q})=P} \mathcal{E}\left(\log\left(1 + \frac{\mathbf{h}^H\mathbf{Q}\mathbf{h}}{\sigma_n^2}\right)\right) \quad (4)$$

where the expectation is over channel state information $\mathbf{h}$ from (3). $tr(\cdot)$ is the trace operator. $\mathbf{Q}$ is the covariance matrix of the transmit signals $\mathbf{x}$ as defined in (2). The capacity can be achieved by a Gaussian codebook with zero mean and covariance matrix $\mathbf{Q^o}$. If the transmitter receives the channel covariance matrix $\mathbf{\Sigma}$ only instead of the concrete channel realization (mean feedback), it does not have any information about the actual attenuation of each transmit-receive pair but possesses directional information regarding the signal subspaces that can be used for beamforming (see [6]).

Regarding the optimal transmit covariance matrix $\mathbf{Q^o}$, the following results are known from [4]: The optimal $\mathbf{Q^o}$ achieving capacity in (4) is given by $\mathbf{Q^o} = \mathbf{U_\Sigma \Lambda}_Q^o \mathbf{U}_\mathbf{\Sigma}^H$ with $\mathbf{\Sigma} = \mathbf{U_\Sigma \Lambda}_\Sigma \mathbf{U}_\mathbf{\Sigma}^H$. $\mathbf{\Lambda}_Q^o$ is a diagonal matrix with eigenvalues $\lambda_1, ..., \lambda_{n_T}$ and $\mathbf{\Lambda}_\Sigma$ is diagonal matrix containing the eigenvalues $\boldsymbol{\mu}= [\mu_1, ..., \mu_{n_T}]$ of Σ. The optimal eigenvalues are found numerically. The beamforming vectors $\mathbf{u}_1, ..., \mathbf{u}_m$ are given by the corresponding eigenvectors of the channel covariance matrix.

Furthermore it was shown that the capacity with known channel covariance matrix at the transmitter is given by

$$C_{opt}^{cfCSI}(P) = \max_{\sum_{i=1}^{n_T} \lambda_i = P} \mathcal{E}\left(\log\left(1 + \rho \sum_{i=1}^{n_T} \mu_i \lambda_i w_i\right)\right) \quad (5)$$

with λ_i as eigenvalues of the transmit covariance matrix $\mathbf{Q}$ and with random w_i white independent distributed according to standard exponential distribution and $\rho = 1/\sigma_n^2$ as SNR. The number of eigenvalues greater than zero correspond to the multiplexing gain of the system. In [7] the authors characterized the multiplexing gain of the MISO system.

3. IMPACT OF CORRELATION ON ERGODIC CAPACITY

In this section, we present the complete characterization of the impact of correlation on the ergodic capacity of MISO systems with different levels of CSI at the transmitter. We consider the cases in which the transmitter has no CSI, perfect CSI and covariance feedback. We use the theory of majorization [8] in order to analyze the impact of correlation. This is the well established approach to model the correlation in MIMO systems [9]. One says that a vector majorizes another if the sum of the m largest values of this vector is greater than or equal to the sum of the m largest values of the other.

3.1. No CSI at the transmitter

The ergodic capacity without CSI at the transmitter is given by

$$C_{opt}^{noCSI}(\boldsymbol{\mu}) = \mathcal{E}\log(1 + \frac{\rho}{n_T}\sum_{l=1}^{n_T} \mu_l w_l). \quad (6)$$

In order to characterize the impact of correlation on the ergodic capacity in (6) we have the following theorem:

Theorem 1: For arbitrary eigenvalue vectors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ we have the following implication

$$\boldsymbol{\mu}^1 \succ \boldsymbol{\mu}^2 \Longrightarrow C_{opt}^{noCSI}(\boldsymbol{\mu}^1) \leq C_{opt}^{noCSI}(\boldsymbol{\mu}^2). \quad (7)$$

i.e. the capacity of the single user MISO system with uninformed transmitter is Schur-concave.

Proof: The proof can be found in [10].

3.2. Perfect CSI at the transmitter

The ergodic capacity with perfect CSI at the transmitter is given by

$$\begin{aligned} C_{opt}^{pCSI} &= \mathcal{E}\log(1 + \rho||\mathbf{h}||^2) \quad (8)\\ &= \mathcal{E}\log(1 + \rho\sum_{l=1}^{n_T} \mu_l w_l). \end{aligned}$$

Remark: For the capacity with perfect CSI (9), we obtain the same term as for the capacity without CSI (6) with the substitution $\rho = \tilde{\rho}/n_T$. In the next theorem, we state that the ergodic capacity is again Schur-concave with respect to the correlation properties $\boldsymbol{\mu}$.

Theorem 2: For arbitrary eigenvalue vectors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ we have the following implication

$$\boldsymbol{\mu}^1 \succ \boldsymbol{\mu}^2 \Longrightarrow C_{opt}^{pCSI}(\boldsymbol{\mu}^1) \leq C_{opt}^{pCSI}(\boldsymbol{\mu}^2). \quad (9)$$

i.e. the capacity of the single user MISO system with perfectly informed transmitter is Schur-concave.

Proof: By using the same arguments as in the uninformed case we can follow that the ergodic capacity is Schur-concave.

3.3. Covariance Feedback

The ergodic capactity with covariance feedback is given in (4). The next theorem states that the ergodic capacity is Schur-convex with respect to the correlation vector $\boldsymbol{\mu}$.

Theorem 3: For arbitrary eigenvalue vectors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ we have the following implication

$$\boldsymbol{\mu}^1 \succ \boldsymbol{\mu}^2 \Longrightarrow C_{opt}^{cfCSI}(\boldsymbol{\mu}^1) \geq C_{opt}^{cfCSI}(\boldsymbol{\mu}^2). \quad (10)$$

i.e. the capacity of the single user MISO system with covariance feedback is Schur-convex.

Proof: The proof can be found in [11].

3.4. Characterization of the optimum power allocation for covariance feedback

In order to characterize the optimum power allocation we define the power vector $\mathbf{p} = [p_1, ..., p_{n_T}]$ with $p_i = \lambda_i$ with the sum power constraint $||\mathbf{p}|| \leq P$. For fixed channel eigenvalues μ_k, the ergodic capacity is a function of the power allocation and it follows from (5)

$$C^{cfCSI}(\mathbf{p}, \rho, \boldsymbol{\mu}) = \mathcal{E}\log(1 + \rho\sum_{k=1}^{n_T} p_k \mu_k w_k). \quad (11)$$

The maximum capacity if given by

$$C_{opt}^{cfCSI}(\rho,\boldsymbol{\mu}) = \max_{||\mathbf{p}||_1\leq 1} C^{cfCSI}(\mathbf{p},\rho,\boldsymbol{\mu}). \quad (12)$$

Furthermore, we define the following coefficients

$$\alpha_k(\mathbf{p}) = \int_0^\infty e^{-t} \prod_{l=1,l\neq k}^{n_T} \frac{1}{1+\rho t p_l \mu_l} \frac{\rho\mu_k}{(1+\rho t\mu_k p_k)^2} dt. \quad (13)$$

Finally, we define the set of indices for which a given power allocation has entries greater than zero

$$\mathcal{I}(\hat{\mathbf{p}}) = \{k \in [1...n_T] : \hat{p}_k > 0\}. \quad (14)$$

The following theorem provides a characterization of the power allocation $\hat{\mathbf{p}}$ which maximizes the expression in (12).

Theorem 4: A necessary and sufficient condition for the optimality of a power allocation $\hat{\mathbf{p}}$ is

$$\{k_1, k_2 \in \mathcal{I}(\hat{\mathbf{p}}) \implies \alpha_{k_1} = \alpha_{k_2} \text{ and} \quad (15)$$

$$k \notin \mathcal{I}(\hat{\mathbf{p}}) \iff \alpha_k \leq \max_{l\in\mathcal{I}(\hat{\mathbf{p}})} \alpha_l\}. \quad (16)$$

This means that all indices l which obtain some power p_l greater than zero have the same $\alpha_l = \max_{l\in[1...n_T]}$. Furthermore, all other α_i are less or equal to α_l.

Proof: The proof can be found in [11].

3.5. Algorithm for optimum power allocation with covariance feedback

We use the Theorem 1 from the last section to provide the following algorithm. This computes the optimum power allocation for the MISO system with covariance feedback (algorithm 1).

Algorithm 1 Optimum power allocation

Require: given $\boldsymbol{\mu}$ and SNR ρ
  $\boldsymbol{\lambda}^1 = [1, 0, ..., 0]$
  for $i = 1$ to $n_T - 1$ **do**
    if $\alpha_i(\boldsymbol{\lambda}^i) \geq \alpha_{i+1}(\boldsymbol{\lambda}^i)$ **then**
      optimum solution is given in $\boldsymbol{\lambda}^i$
    else
      find $\boldsymbol{\lambda}^i + 1$ with $\alpha_1(\boldsymbol{\lambda}^{i+1}) = ... = \alpha_{i+1}(\boldsymbol{\lambda}^{i+1})$
    end if
  end for
Ensure: $\sum_{k=1}^{n_T} \lambda_k = 1$

We start with the beamforming solution in $\boldsymbol{\lambda}^1$ and check whether the condition in (16) is fulfilled. If it is not fulfilled we split the transmission power to direction one and two ($\boldsymbol{\lambda}^2$) in such a way that $\alpha_1(\boldsymbol{\lambda}^1) = \alpha_2(\boldsymbol{\lambda}^1)$. Next, we check again the condition in (16) for $\boldsymbol{\lambda}^2$ and so on.

For the algorithmic implementation, we mention that the integral in (13) can be written for the case that $\lambda_m = \lambda_{m+1} = ... = \lambda_{n_T} = 0$ as

$$\alpha_m(\lambda) = \rho\mu_m \int_0^\infty e^{-t} \frac{1}{\prod_{l=1}^{m-1}(1+\rho t\lambda_l\mu_l)} dt.$$

3.6. Relationship between the different CSI schemes

The inequality chain in the next corollary shows the relation between the different CSI schemes and different levels of correlation. Assume that the correlation vector $\boldsymbol{\mu}^1$ majorizes $\boldsymbol{\mu}^2$, i.e. $\boldsymbol{\mu}^1 \succ \boldsymbol{\mu}^2$. We define the fully correlated vector $\boldsymbol{\psi} = [1, 0, ..., 0]^T$ and the completely uncorrelated vector as $\boldsymbol{\chi} = [1/n_T, 1/n_T, ..., 1/n_T]^T$.

Corollary 1: We have for the ergodic capacity in MISO systems with different levels of correlation and different CSI at the transmitter the following inequalities:

$$C_{opt}^{noCSI}(\boldsymbol{\psi}) \leq C_{opt}^{noCSI}(\boldsymbol{\mu}^2) \leq C_{opt}^{noCSI}(\boldsymbol{\mu}^1) \leq C_{opt}^{noCSI}(\boldsymbol{\chi}) =$$
$$C_{opt}^{cfCSI}(\boldsymbol{\chi}) \leq C_{opt}^{cfCSI}(\boldsymbol{\mu}^1) \leq C_{opt}^{cfCSI}(\boldsymbol{\mu}^2) \leq C_{opt}^{cfCSI}(\boldsymbol{\psi}) =$$
$$C_{opt}^{pCSI}(\boldsymbol{\psi}) \leq C_{opt}^{pCSI}(\boldsymbol{\mu}^2) \leq C_{opt}^{pCSI}(\boldsymbol{\mu}^1) \leq C_{opt}^{pCSI}(\boldsymbol{\chi}). \quad (17)$$

Proof: The inequalities follow from (7), (9) and (10).

The worst case scenario is the uninformed transmitter with fully correlated channels $C_{opt}^{noCSI}(\boldsymbol{\psi})$. In the best case the perfectly informed transmitter with completely uncorrelated channels is $C_{opt}^{pCSI}(\boldsymbol{\chi})$.

In figure (1), we show the capacity gain for a specific scenario with two transmit antennas $n_T = 2$ over the correlation coefficient μ_1 at a SNR of 20 dB. In figure (1), we illustrate the

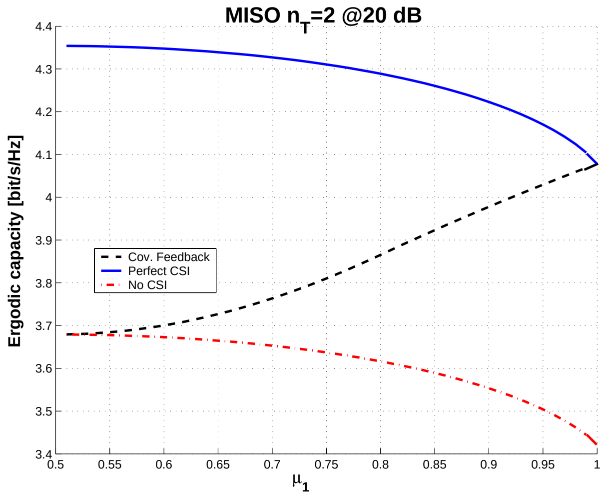

Fig. 1. Capacity as a function of correlation for MISO 2×1 system with different levels of CSI

inequality chain from (17). We observe that the capacity for this scenario can be increased by more CSI. Furthermore, it can be increased by higher or smaller correlation. Starting with no CSI from 3,44 bit/s/Hz (100 %) the capacity increases with less correlation to 3.68 bit/s/Hz (107 %). With covariance feedback we start uncorrelated at 3.68 bit/s/Hz and increase the capacity up to 4.08 bit/s/Hz (119 %) with more correlation. In the scenario with perfect CSI, we start at this value (4.08 bit/s/Hz) completely correlated and gain up to 4.35 bit/s/Hz (126 %) with less correlation.

3.7. Discussion

We have proven that the impact of the correlation on the ergodic capacity strongly depends on the level of CSI at the transmitter.

If we do not have any CSI at the transmitter, the optimum strategy is the equal split of the transmit power. Therefore, we cannot adjust to a potential correlation in the channel. Hence, we waste transmission power by allocating power for channels with small eigenvalues. In the case in which we have covariance knowledge at the transmitter, we can adjust the transmission strategy to the correlation properties of the channel and can take advantage of the correlation. Therefore, the correlation helps increasing the ergodic capacity. In the case in which we have perfect CSI at the transmitter, we know each channel realization at the transmitter and can apply the optimum transmission strategy at each time point. Therefore, the advantage of having a correlated channel with distinct average directions and transmission powers vanishes and the ergodic capacity is at its highest for uncorrelated channels.

4. NUMERICAL SIMULATION

In figure (2), we present the optimum power allocation for a MISO system with two transmit antennas for correlation $\mu_1 = [0.5, ..., 1]$ and SNR from 0 dB up to 50 dB.

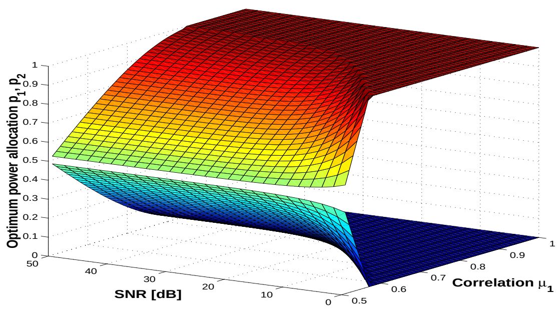

Fig. 2. optimum power allocation for two transmit antenna MISO system with covariance feedback

In figure (3), we show the ergodic capacity of a 3×1 MISO system at a SNR of 20 dB for as a function of the correlation properties. The eigenvalues are sorted $\boldsymbol{\mu}_1 \geq \boldsymbol{\mu}_2 \geq \boldsymbol{\mu}_3$ and their sum is equal to one, i.e. $\sum_{i=1}^{3} \mu_i = 1$. The largest channel covariance matrix eigenvalue $\boldsymbol{\mu}_1$ ranges from 0.33 to 1. And the second largest eigenvalue $\boldsymbol{\mu_2}$ is varied from $1 - \boldsymbol{\mu}_1$ down to $(1 - \boldsymbol{\mu}_1)/2$.

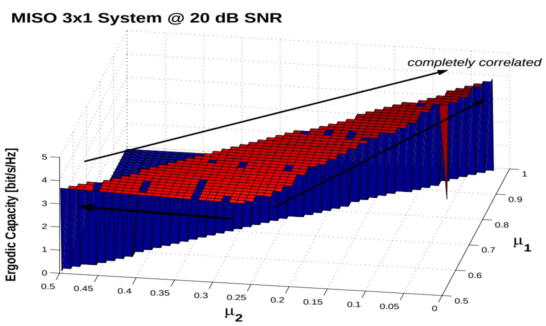

Fig. 3. Ergodic capacity for three transmit antenna MISO system at SNR 20 dB

In figure (3), the arrows indicate the direction of the ergodic capacity increase. The upper right point corresponds with the case in which the channel is completely correlated, i.e. $\boldsymbol{\mu} = [1, 0, 0]^T$. At this point the ergodic capacity decreases in all directions. The point at which the channel is completely uncorrelated is $\boldsymbol{\mu} = [1/3, 1/3, 1/3]^T$. It is in the front side in the middle. At this point, the ergodic capacity is at its minimum.

5. CONCLUSION

In this work, we studied the optimum power allocation and the impact of correlation on the ergodic capacity of a MISO system with covariance feedback. We characterized the optimum power allocation in terms of a necessary and sufficient condition for the optimality. As a result, we developed a computational efficient algorithm which computes the optimum power allocation for given channel covariance matrix and SNR. Furthermore, we studied the impact of correlation on the ergodic capacity. Recently, it was shown that the ergodic capacity in a MISO system without CSI at the transmitter is Schur-concave with respect to the covariance matrix eigenvalues. We showed that this is the other way around for MISO systems with covariance feedback, i.e. the ergodic capacity is Schur-convex. Furthermore, we proved that the ergodic capacity of the MISO system with perfect CSI at the transmitter is Schur-concave with respect to the correlation properties. We provided an inequality which related the ergodic capacities of different CSI schemes and different correlation properties.

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