SUBSPACE ALGORITHMS FOR ERROR LOCALIZATION WITH DFT CODES

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ABSTRACT

In this paper, we propose two subspace algorithms for error localization with quantized DFT codes. The algorithms are similar to the MUSIC and the minimum-norm algorithms followed in the array signal processing for direction of arrival (DOA) estimation. We present the algorithms in a generalized form with a variable dimension syndrome covariance matrix. Simulation results show that their localization performances are similar, and they achieve the peak values when the rank of the covariance matrix reaches the maximum value. They also perform better than the coding theoretic approach over a broad range of channel error to quantization noise ratio.

1. INTRODUCTION

This paper addresses the subject of channel error correction with DFT codes [1–4]. DFT codes are being considered for joint source-channel coding in order to provide robustness to data loss or corruption in communication channels [3, 4]. It is known that DFT codes are cyclic codes in the complex field [1]. However, the code properties do not hold once the codevectors are quantized. The error correction problem becomes analogous to the problem of estimation of directions and amplitudes of plane waves (DOAs) incident on a uniform linear array [5, 6].

An (N, K) DFT code is a linear block code whose generator matrix consists of any K columns from the inverse DFT matrix of order N [1]. A parity check matrix of the code consists of the remaining $N - K \equiv d$ columns of the inverse DFT matrix. Since the DFT matrix is Hermitian, every codevector misses the parity frequencies. The presence of any error in a codevector is indicated by the parity frequencies being nonzero, which are also known as the syndrome frequencies.

DFT codes are cyclic codes in the complex field [1]. Within the class of DFT codes, there exist BCH codes in the real field and the complex field. A BCH DFT code is an MDS code. An (N, K) DFT code which is an MDS code in the complex field or the real field has minimum hamming distance $d+1$ [2]. Therefore it can correct up to $\lfloor d/2 \rfloor$ sample errors and recover up to d sample erasures.

In this paper, we present two subspace algorithms to localize channel errors which are derived along the lines of the subspace based approaches to DOA estimation. The basic idea of a subspace approach to error localization is to partition a vector space into a channel error subspace and its orthogonal complement, the noise subspace. In the first approach, which we call MUSIC-like because of its similarity with the MUSIC algorithm, the common roots of the polynomials associated with the noise subspace eigenvectors determine the error locations. In the second approach, the roots of the polynomial which is associated with the minimum-norm vector lying in the noise subspace determine the error locations. We present the algorithms for both unquantized and quantized codevectors, and compare their performances with the coding theoretic approach.

2. ERROR LOCALIZATION WITHOUT QUANTIZATION

Let $\mathbf{r}$ denote the received vector when the transmitted codevector $\mathbf{y}$ is corrupted by the error vector $\mathbf{e}$. Then

$$\mathbf{r} = \mathbf{y} + \mathbf{e}. \tag{1}$$

The syndrome of $\mathbf{r}$ is given as

$$\mathbf{s} = H^h \mathbf{r} = H^h(\mathbf{y} + \mathbf{e}) = H^h \mathbf{e}, \tag{2}$$

where H denotes the parity check matrix and $\mathbf{s} \equiv [s(1), \ldots, s(d)]^t$ is a column vector of length d. The superscript t denotes the matrix transposition operation.

Let d be equal to $2l$ or $2l+1$ for some positive integer l if it is even or odd respectively. Let the received samples contain ν errors where $\nu \leq l$. Let $i_1, i_2, \ldots, i_\nu$ denote the indices of the erroneous samples. Let $X_k \equiv e^{-2\pi j i_k/N}$, $k = 1, \ldots, \nu$. Let $V_e^{(m)}$ denote the error locator matrix whose ith column is $[1, X_i, \ldots, X_i^{m-1}]^t$, $i = 1, \ldots, \nu$. The superscript (m) will refer to the number of rows of $V_e^{(m)}$. The number of columns in $V_e^{(m)}$ is equal to the number of errors. We will refer to the columns of $V_e^{(m)}$ as the error locator vectors of order m. Since the roots are distinct, the error locator vectors are linearly independent when $m \geq \nu$. When $m > \nu$, they define a ν-dimensional subspace of the m-dimensional vector space, which we will refer to as the channel error subspace. The orthogonal complement of this subspace has dimension $m-\nu$, and we will refer to it as the noise subspace.

Let us assume that $\nu + 1 \leq m \leq d - \nu + 1$. Let S_m denote the syndrome matrix defined as

$$S_m \equiv \begin{bmatrix} s(1) & s(2) & \ldots & s(d-m+1) \\ s(2) & s(3) & \ldots & s(d-m+2) \\ \cdots & \cdots & \cdots & \cdots \\ s(m) & s(m+1) & \ldots & s(d) \end{bmatrix}.$$

Let $R_m \equiv \frac{1}{d-m+1} S_m S_m^h$. We will refer to R_m as the syndrome covariance matrix. It can be shown that the rank of R_m is equal to the number of errors, ν (see Appendix). Therefore it has ν nonzero eigenvalues. The eigen-decomposition of R_m is given as

$$R_m = [U_e^{m\times\nu} \; U_n^{m\times(m-\nu)}] \begin{bmatrix} \Delta_e^{\nu\times\nu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}^{(m-\nu)\times(m-\nu)} \end{bmatrix} [U_e \; U_n]^h,$$

where Δ_e contains the nonzero eigenvalues. Now we have the following proposition.

Proposition 1 *The columns of U_e span the channel error subspace.*

Proof: See Appendix.

Because of this result, the error locator vectors are orthogonal to the eigenvectors of U_n, which span the noise subspace. That is, $V_e^{(m)h} U_n = \mathbf{0}_{\nu \times (m-\nu)}$. Let $\mathbf{u}_i$ denote the ith column of U_n. Let $\Phi_i(x)$ denote the polynomial whose coefficients are equal to the components of $\mathbf{u}_i$. Then the above equation can be written as

$$u_{i,1} + u_{i,2} X_k^{-1} + \cdots + u_{i,m} X_k^{-(m-1)} = 0, \quad k = 1, \ldots, \nu;\ i = 1, \ldots, m - \nu. \tag{3}$$

This shows that $X_1, \ldots,, X_\nu$ are common roots of $\Phi_1(x), \ldots, \Phi_{m-\nu}(x)$. Further, since $\nu < m$, they are the only common roots. When $m = \nu + 1$, the noise subspace has dimension one, and the roots of $U_n \equiv \mathbf{u}_1$ are $X_1, \ldots, X_\nu$.

Since the noise subspace is orthogonal to the error subspace, any linear combination of the columns of U_n is orthogonal to the error locator vectors. This can be equivalently said as any linear combination of $\Phi_i(x)$'s has roots at $X_1, \ldots, X_\nu$. Let $\mathbf{a}$ be a noise subspace vector whose first element is 1, that is, $\mathbf{a} \equiv [1\ \mathbf{w}^t]^t$ where $\mathbf{w}$ denotes the vector of remaining elements. Let us partition U_e as $U_e \equiv \begin{bmatrix} \mathbf{p}_e \\ P_e \end{bmatrix}$ where $\mathbf{p}_e$ denotes the first row of U_e. Similarly let us partition U_n as $U_n \equiv \begin{bmatrix} \mathbf{p}_n \\ P_n \end{bmatrix}$ where $\mathbf{p}_n$ denotes the first row of U_n. Since $\mathbf{a}$ is orthogonal to the columns of U_e, we get $[\mathbf{p}_e^h\ P_e^h][1\ \mathbf{w}^t]^t = \mathbf{0}$. This gives

$$\mathbf{p}_e^h + P_e^h \mathbf{w} = \mathbf{0} \quad \text{or,} \quad P_e^h \mathbf{w} = -\mathbf{p}_e^h. \tag{4}$$

This set of equations has more unknowns than the number of equations except in the special case when $m = \nu + 1$ (P_e is square in that case.). Therefore there is no unique solution if $m > \nu + 1$. However, we can find the minimum-norm solution for $\mathbf{w}$ ($\|\mathbf{w}\|^2$ is minimum), which is given as

$$\mathbf{w}_m = -(P_e^h)^+ \mathbf{p}_e^h. \tag{5}$$

Here $(P_e^h)^+$ denotes the pseudo-inverse of P_e^h. Using the formula for the pseudo-inverse and the orthogonality between the columns of U_e and U_n, this expression can be simplified as [6]

$$\mathbf{w}_m = \frac{P_n \mathbf{p}_n^h}{\mathbf{p}_n \mathbf{p}_n^h}. \tag{6}$$

Since $\|\mathbf{a}\|^2 = 1 + \|\mathbf{w}\|^2$, $\mathbf{a}_m \equiv [1\ \mathbf{w}_m^t]^t$ is the minimum-norm solution of $\mathbf{a}$. Because of the minimum norm property, $\mathbf{a}_m$ is unique. Let $A_m(x) \equiv a_m(0) + a_m(1)x^{-1} + \cdots + a_m(m-1)x^{-(m-1)}$. Because $\mathbf{a}_m$ lies in the noise subspace, $X_1, \ldots, X_\nu$ are the roots of $A_m(x)$. However, since the degree of $A_m(x)$ is $m-1$, it also has $m-1-\nu$ other roots. It has been shown that $X_1, \ldots, X_\nu$ are the only roots which lie on the unit circle in the complex plane [7]. Note that X_i's are nothing but N-th roots of unity. Therefore the locations of the errors can be known by finding the roots of $A_m(x)$ over the N-th roots of unity.

Once the error locations are known, the error values can be determined by solving the first ν syndrome equations in Eqn. 2.

3. ERROR LOCALIZATION WITH QUANTIZATION

The transmission of the codevectors in digital form requires all codevectors to be quantized. As a result, every codevector contains N sample errors irrespective of any channel error. Therefore the error correction algorithms based on the coding theory principle cannot be applied directly. The problem of error correction now becomes a problem of estimation. The decoding algorithm aims at localizing and finding the channel errors having large magnitudes compared to the quantization noise.

Let $\mathbf{q}$ denote the quantization noise of codevector $\mathbf{y}$. With channel error $\mathbf{e}$, the received vector is given as $\hat{\mathbf{r}} = \mathbf{y} + \mathbf{q} + \mathbf{e}$. We will denote the terms defined earlier with a hat to indicate the presence of quantization noise. Therefore the syndrome is given as

$$\hat{\mathbf{s}} = H^h \hat{\mathbf{r}} = H^h \mathbf{q} + H^h \mathbf{e} = \mathbf{s}_q + \mathbf{s}_e, \tag{7}$$

where $\mathbf{s}_q \equiv H^h \mathbf{q}$ and $\mathbf{s}_e \equiv H^h \mathbf{e}$. It is easy to see that $\mathbf{s}_q$ denotes the contribution of the quantization noise to the syndrome. Therefore a nonzero syndrome does not imply the presence of channel errors.

Let us assume that the quantization noise $\mathbf{q}$ is white and is uncorrelated with the channel errors. Each component of $\mathbf{q}$ is assumed to have mean zero and variance σ^2. The channel error magnitudes are assumed to be large compared to σ^2. Since $\hat{\mathbf{s}} = \mathbf{s}_e + \mathbf{s}_q$, the syndrome matrix $\hat{S}_m$ can be expressed as $\hat{S}_m = S_{me} + S_{mq}$ where S_{me} denotes the part due to the channel errors and S_{mq} denotes the part due to the quantization noise. Now $\hat{R}_m = \frac{1}{d-m+1} \hat{S}_m \hat{S}_m^h$. Expanding $\hat{S}_m \hat{S}_m^h$, we get

$$\begin{aligned} \hat{R}_m &= R_m + \frac{1}{d-m+1}(S_{mq} S_{mq}^h + S_{me} S_{mq}^h + S_{mq} S_{me}^h) \\ &\equiv R_m + R_{mn}, \end{aligned} \tag{8}$$

where R_{mn} denotes the noise term on the right hand side. The presence of R_{mn} will perturb the eigenvectors and the eigenvalues of R_m. The statistical behaviour of this perturbation depends on the statistical properties of R_{mn}. Since $\mathbf{q}$ is assumed to be white and uncorrelated with $\mathbf{e}$, $\mathbb{E}(S_{me} S_{mq}^h) = \mathbb{E}(S_{mq} S_{me}^h) = \mathbf{0}$, and $\mathbb{E}(S_{mq} S_{mq}^h) = \sigma^2 (d - m + 1) I_m$, where $\mathbb{E}$ denotes the mathematical expectation operator and I_m denotes the identity matrix of order m. Therefore $\mathbb{E}(R_{mn}) = \sigma^2 I_m$. This shows that the expected eigenvalues of $\hat{R}_m$ associated with the error subspace are the diagonal elements of $\Delta_e + \sigma^2 I_\nu$, and the expected eigenvalues associated with the noise subspace are σ^2. Further, the expectation of the perturbation of $\hat{U}_e$ is zero.

The number of errors can be estimated from the distribution of the eigenvalues. Gabay and Duhamel [4] estimate the number of errors as the number of eigenvalues greater than $\beta\sigma^2$, where β is set empirically. We have observed that at high channel error to quantization noise ratio, this approach performs much better than the Akaike information criterion (AIC) and the minimum description length criterion (MDL), the two well known information theoretic criteria in array processing for estimating the number of DOAs.

The eigen-decomposition of $\hat{R}_m$ gives

$$\hat{R}_m = [\hat{U}_e^{m \times \nu}\ \hat{U}_n^{m \times (m-\nu)}] \begin{bmatrix} \hat{\Delta}_e^{\nu \times \nu} & \mathbf{0} \\ \mathbf{0} & \hat{\Delta}_n^{(m-\nu) \times (m-\nu)} \end{bmatrix} [\hat{U}_e\ \hat{U}_n]^h,$$

where $\hat{\Delta}_e$ contains the ν largest eigenvalues. The columns of $\hat{U}_e$ span the estimated error subspace, and the columns of $\hat{U}_n$ span the estimated noise subspace. The errors can be localized by minimizing the following function over the Nth roots of unity:

$$E_m(x) = \mathbf{v}_x^{(m)h} \hat{U}_n \hat{U}_n^h \mathbf{v}_x^{(m)}, \tag{9}$$

where $\mathbf{v}_x^{(m)} \equiv [1, x, x^2, \ldots, x^{m-1}]^t$. Alternatively, since $\hat{U}_e\hat{U}_e^h + \hat{U}_n\hat{U}_n^h = I_m$, $\mathbf{v}_x^{(m)h}\hat{U}_e\hat{U}_e^h\mathbf{v}_x^{(m)}$ can be maximized over the roots of unity. The above algorithm is similar to the MUSIC algorithm followed in array signal processing for DOA estimation [6].

The minimum-norm estimate can be obtained from $\hat{U}_n$. If $\hat{U}_n \equiv \begin{bmatrix}\hat{\mathbf{p}}_n\\ \hat{P}_n\end{bmatrix}$, then

$$\hat{\mathbf{a}}_m = \begin{bmatrix} 1 \\ \frac{\hat{P}_n\hat{\mathbf{p}}_n^h}{\hat{\mathbf{p}}_n\hat{\mathbf{p}}_n^h}\end{bmatrix}. \tag{10}$$

The errors are localized by minimizing $|\hat{A}_m(x)|^2$ over the Nth roots of unity.

Once the error locations are known, the error values can be estimated by solving $H^h\hat{\mathbf{e}} = \hat{s}$ in the least square sense.

4. PERFORMANCE ANALYSIS

The aim of this section is to compare the localization performances of the coding theoretic approach and the minimum-norm approach. The comparison is made in terms of the perturbation of the channel error subspace.

In the coding theoretic approach, the errors are localized using the concept of an error locator polynomial [2] which is defined as

$$\Lambda(x) \equiv \prod_{i=1}^{\nu}(1 - X_i x^{-1}) = \Lambda_0 + \Lambda_1 x^{-1} + \ldots + \Lambda_\nu x^{-\nu}, \tag{11}$$

where $\Lambda_0 = 1$. The coefficients $\Lambda_1, \ldots, \Lambda_\nu$ are found by solving the following set of convolution equations [2]:

$$s(i)\Lambda_\nu + s(i+1)\Lambda_{\nu-1} + \cdots + s(i+\nu)\Lambda_0 = 0, \quad i = 1, \ldots, d-\nu. \tag{12}$$

Let $\mathbf{\Lambda}_m \equiv [1, \Lambda_1, \ldots, \Lambda_\nu, \overbrace{0, \ldots, 0}^{m-\nu-1}]^t$. Clearly, $\Lambda(x)$ is the polynomial associated with $\mathbf{\Lambda}_m$. Since $X_1, \ldots, X_\nu$ are the roots of $\Lambda(x)$, $V_e^{(m)h}\mathbf{\Lambda}_m = \mathbf{0}$. This implies that $\mathbf{\Lambda}_m$ lies in the noise subspace. Therefore $U_e^h\mathbf{\Lambda}_m = \mathbf{0}$. $\mathbf{\Lambda}_m$ can be partitioned as $[1 \;\; \mathbf{\Lambda}^t \;\; \mathbf{0}_{1\times(m-\nu-1)}]^t$, where $\mathbf{\Lambda} \equiv [\Lambda_1 \;\; \Lambda_2 \;\; \ldots, \Lambda_\nu]^t$. Let us partition U_e as $U_e = \begin{bmatrix}\mathbf{p}_e\\ P_{e_1}\\ P_{e_2}\end{bmatrix}$ where $\mathbf{p}_e$ denotes the first row as defined before, P_{e_1} denotes the next ν rows and $\mathbb{P}_{e_2}$ denotes the remaining rows. Using the partitioned forms of U_e and $\mathbf{\Lambda}_m$, we get

$$\mathbf{p}_e^h + P_{e_1}^h\mathbf{\Lambda} = \mathbf{0}, \quad \text{or} \quad P_{e_1}^h\mathbf{\Lambda} = -\mathbf{p}_e^h. \tag{13}$$

Taking the differential of both sides, and simplifying, we get

$$\delta\mathbf{\Lambda} \approx -(P_{e_1}^h)^{-1}\delta U_e^h\mathbf{\Lambda}_m. \; (\because P_{e_1} \text{ is invertible}) \tag{14}$$

The perturbation of the error subspace is directly proportional to the projection of $\delta\mathbf{\Lambda}_m \equiv [0 \;\; \delta\mathbf{\Lambda}^t \;\; \mathbf{0}_{1\times(m-\nu-1)}]^t$ onto the error subspace. Let $\delta\mathbf{\Lambda}_{me}$ denote the projection. Then $\|\delta\mathbf{\Lambda}_{me}\|^2 = \mathbf{\Lambda}_m^h\delta U_e\delta U_e^h\mathbf{\Lambda}_m$. Therefore

$$\mathbb{E}(\|\delta\mathbf{\Lambda}_{me}\|^2) = \mathbf{\Lambda}_m^h\mathbb{E}(\delta U_e\delta U_e^h)\mathbf{\Lambda}_m. \tag{15}$$

Using the asymptotic formula for $\mathbb{E}(\delta U_e\delta U_e^h)$ from [8], and simplifying, we get

$$\mathbb{E}(\|\delta\mathbf{\Lambda}_{me}\|^2) = \|\mathbf{\Lambda}_m\|^2\frac{\sigma^2}{d-m+1}\sum_{i=1}^{\nu}\frac{\lambda_i}{(\lambda_i-\sigma^2)^2}, \quad \nu+1 \le m \le l+1, \tag{16}$$

where λ_i's are the ν largest eigenvalues of $\hat{R}_m$.

Consider the minimum-norm algorithm. Let $\delta\mathbf{a}_m$ denote the perturbation of $\mathbf{a}_m$ due to the quantization. Following exactly similar steps as above, we can obtain [8]

$$\mathbb{E}(\|\delta\mathbf{a}_{me}\|^2) = \|\mathbf{a}_m\|^2\frac{\sigma^2}{d-m+1}\sum_{i=1}^{\nu}\frac{\lambda_i}{(\lambda_i-\sigma^2)^2}, \quad \nu+1 \le m \le l+1, \tag{17}$$

where $\delta\mathbf{a}_{me}$ is the projection of $\delta\mathbf{a}_m$ onto the error subspace. Since $\mathbf{a}_m$ is the minimum-norm solution, $\|\mathbf{a}_m\|^2 \le \|\mathbf{\Lambda}_m\|^2$. This implies that $\mathbb{E}(\|\delta\mathbf{a}_{me}\|^2) \le \mathbb{E}(\|\delta\mathbf{\Lambda}_{me}\|^2)$. It can be shown that the norm of $\mathbf{a}_m$ is a decreasing function of m. This means that the expected perturbation with the minimum-norm algorithm is a decreasing function of m.

5. SIMULATION RESULTS AND CONCLUSION

In order to test and validate the algorithms, we performed simulations over a Gauss-Markov source with mean 0, variance 1, and correlation coefficient 0.9. The source was encoded with a $(18, 9)$ DFT code and quantized with a 4-bit uniform scalar quantizer. First we simulated the subspace algorithms for different values of m for a given number of channel errors. Fig. 1 and Fig. 2 show the relative frequencies of correct localization for one and two channel errors respectively. We observe that both the MUSIC-like and the minimum-norm methods perform similarly. The performance is the best when m is equal to 5, i.e., the syndrome matrix has dimension 5×5. We have observed that increasing m further degrades the performances for multiple errors. Note that increasing m increases the dimensionality of the noise subspace, however, since the number of columns of S_m decreases with the increase in m, the syndrome covariance matrix has increasing number of zero eigenvalues. This means that the performance is the best when the rank of the covariance matrix reaches the highest possible value. Fig. 3 and Fig. 4 compare the localization performances of the coding theoretic approach and the subspace approaches with the syndrome matrix dimension 5×5. The performance improvement over the coding theoretic approach is evident.

From the above results we conclude that the MUSIC-like and the minimum-norm subspace approaches have similar localization performances. They also outperform the coding theoretic approach when the dimension of the syndrome matrix is chosen such that the resulting covariance matrix has the maximum rank.

APPENDIX

S_m can be expressed as $S_m = V_e^{(m)}D_eV_e^{(d-m+1)t}$, where D_e is a $\nu\times\nu$ diagonal matrix. Therefore, $R_m = \frac{1}{d-m+1}V_e^{(m)}D_e \times V_e^{(d-m+1)t}V_e^{(d-m+1)*}D_e^hV_e^{(m)h}$. Since $X_1, \ldots, X_\nu$ are distinct and nonzero, and $\nu < m$, the columns of $V_e^{(m)}$ are l.i. and the rows of $V_e^{(d-m+1)t}$ are l.i. Therefore the rank of R_m is equal to ν. Now R_m can be expressed as $R_m = U_e\Delta_eU_e^h$. Therefore,

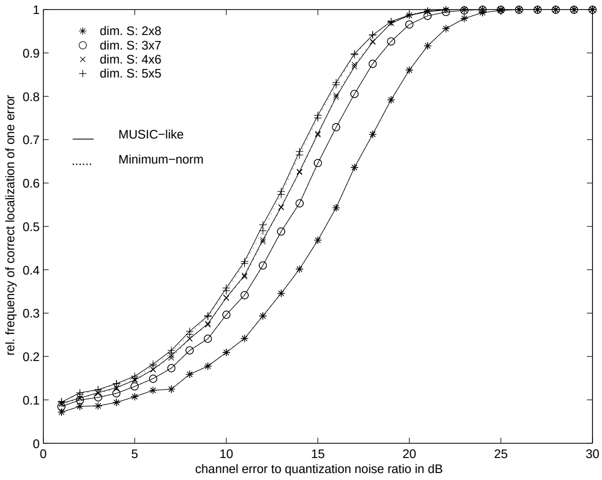

Fig. 1. Relative frequency of correct localization of one error

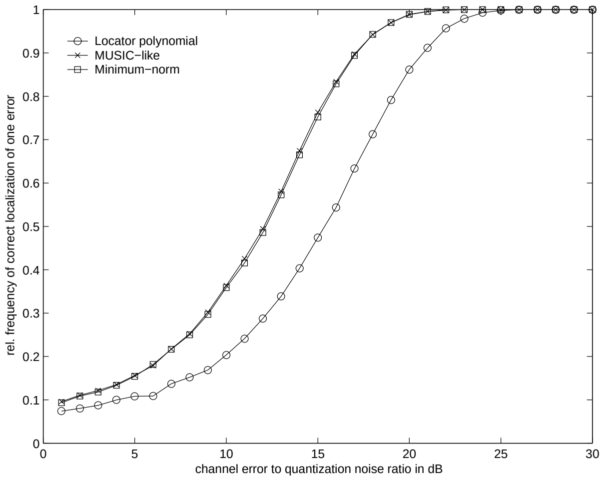

Fig. 3. Relative frequency of correct localization of one error

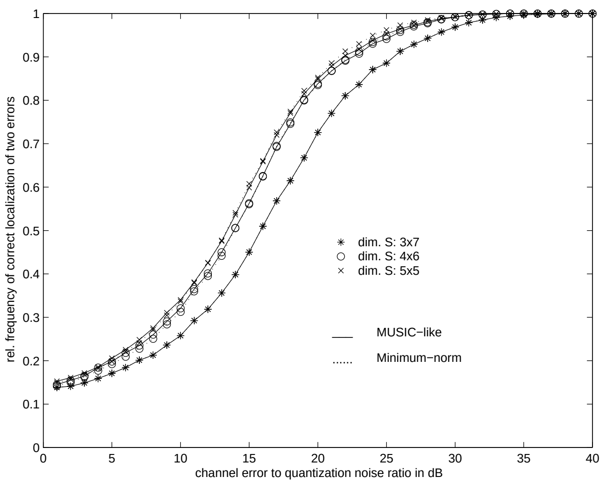

Fig. 2. Relative frequency of correct localization of two errors

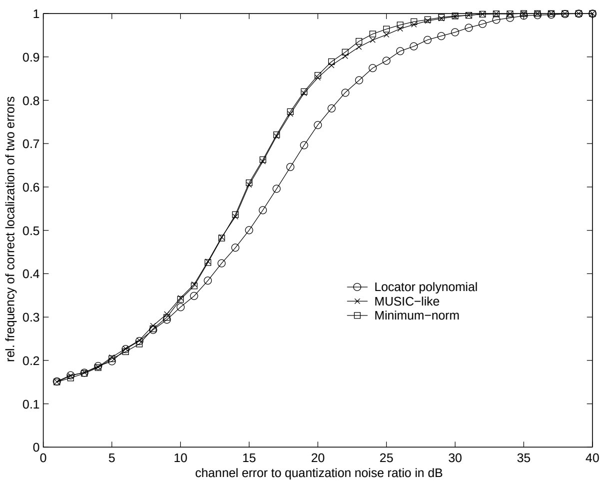

Fig. 4. Relative frequency of correct localization of two errors

$U_e \Delta_e U_e^h = \frac{1}{d-m+1} V_e^{(m)} D_e V_e^{(d-m+1)t} V_e^{(d-m+1)*} D_e^h V_e^{(m)h}$. From this relation, we can express the columns of U_e as linear sums of the columns of $V_e^{(m)}$ and vice versa. Thus the eigenvectors in U_e span the error subspace.

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