QUANTIZED FRAME EXPANSIONS BASED ON TREE-STRUCTURED OVERSAMPLED FILTER BANKS FOR ERASURE RECOVERY

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ABSTRACT

This paper studies quantized frame expansions based on tree-structured oversampled filter banks for robust transmission of multimedia signals over erasure channels. The dependencies between the expansion coefficients introduced by the oversampled tree-structured filter banks are analysed in the z-transform domain. The capacity of correction in the presence of different erasure patterns then follows naturally. This analysis leads to the design of a reconstruction algorithm in the presence of erasures, and first without quantization noise. The impact of quantization error on the reconstruction error is then studied.

For the tree-structured oversampled filter banks, we prove that reconstruction from a larger class of erasure patterns is possible as compared to regular oversampled filter banks. The reconstruction mean square error is significantly lower in the case of tree-structured filter bank as compared to regular oversampled filter banks [6] and block transforms, even if the erasures are bursty in nature.

EDICS: Multiple Descriptive Coding, Joint Source Channel Coding, Erasure Resilient Coding, Oversampled Filter Banks.

1. INTRODUCTION

Data transmission over delivery infrastructures offering no QoS guarantee (e.g. the current best-effort Internet or wireless channels) suffer from impairements due to transients in the physical layer or from losses due to network failure or congestion. In this article we focus on the problem of losses or erasures. Traditional approaches to fight against erasures consist in sending redundant information along with the original information so that the lost data (or at least part of it) can be recovered from the redundant information. The design principles that have prevailed so far stem from Shannon's source and channel separation theorem stating that source and channel optimum performance bounds can be approached as close as desired by designing independently source and channel coding strategies. However, this holds only under asymptotic conditions where both codes are allowed infinite length and complexity. If the design of the system is heavily constrained in terms of complexity or delay, the separate (also called tandem) approach can be largely suboptimal. This observation has recently motivated extensive work on the design of joint source-channel codes. In literature so far, DFT codes have been studied as joint source-channel block codes to obtain robustness to erasures [4]. It is shown in [4] that DFT codes are a special class of frames. Filter bank frame expansions have also been studied to achieve resilience to erasures [1], [3]. In [1], the authors have shown correspondences between oversampled filter banks and frames in $l^2(Z)$. They have shown that if the frames are uniform and tight, the mean square reconstruction error is minimized. The filter banks considered in [1], are however special filter banks which have filters with lengths restricted to the downsampling ratio in order to avoid overlaps, hence are similar to block transforms. Quantized frame expansions based on two-channel oversampled filter banks have been studied in [6]. When used in a tree-structured wavelet signal decomposition only to protect the low frequency bands, the two-channel oversampled filter bank (OFB) allows to recover from a larger range of erasure patterns than block transform codes, such as the DFT codes and the OFB [6]. However, in [6], the reconstruction algorithm exploits only the dependencies introduced in the last stage of the iterated wavelet decomposition. In this article, we extend the results of [6] to general tree-structured oversampled filter banks (TSOFB). The dependencies introduced by these OFB have been analyzed in the z-transform domain. They are expressed in terms of a polynomial in z, $K(z)$, conditioning the existence of perfect reconstruction solutions, when there is no quantization noise. A practical reconstruction algorithm then follows naturally. We prove that a larger class of erasure patterns can be reconstructed when TSOFB are used as compared to OFB. Further, the reconstruction mean square error with quantization for bursty erasure patterns of the TSOFB is significantly lower as compared to OFB and block transforms like the DFT codes.

2. TREE-STRUCTURED OFB

OFB can be constructed from from critically sampled uniform filter banks by replacing the downsampling factor to a number lesser than the number of channels. Consider the tree-structured filter bank shown in Fig. 1, which is critically sampled. Using Noble Identity, this filter bank can be expressed in z-domain as shown in Fig. 2. If the three downsamplers on the last stage have not been used, we arrive at an oversampled tree-structured filter bank shown in Fig. 3. This structure of decomposition allows to support unequal loss protection, the low frequency information being more vital to protect than the high frequency information. The rationale behind the different oversampling ratios associated with the different components is as follows. The coefficients corresponding to the lowpass components, i.e. $X_0(z)$, $X_1(z)$, $X_2(z)$ have to be better protected than the component $X_3(z)$. Amongst $X_0(z)$, $X_1(z)$, $X_2(z)$, more protection is to be given to $X_0(z)$, $X_1(z)$ than $X_2(z)$. If the filters $h_0(n)$ and $h_1(n)$ have length L, then the filters $H_0(z^2)H_0(z)$ and $H_1(z^2)H_0(z)$, denoted as $H_0^{'}(z)$ and $H_1^{'}(z)$ respectively have length $3L-2$. If the filter bank is seen as an encoder, then the rate of the code is $R = \frac{1}{2}$. Let the input sequence $x^{'}(n)$ to the oversampled portion of the filter bank shown in Fig. 3 be written as a vector denoted as $\boldsymbol{x}^{'}$, i.e. $\boldsymbol{x}^{'} = [\cdots\ x^{'}(0)\ x^{'}(1)\ x^{'}(2)\cdots]^t$. Let the three output sequences of the filter bank $x_0(n)$, $x_1(n)$ and $x_2(n)$ be combined in one vector $\boldsymbol{y}$, i.e.

$$\begin{aligned}\boldsymbol{y} = [\cdots\ & x_0(0)\ x_1(0)\ x_2(0)\ x_0(1)\ x_1(1) \\ & x_2(1)\ x_0(2)\cdots]^t. \end{aligned} \qquad (1)$$

We can express $\boldsymbol{y}$ as the output of a linear transform $\boldsymbol{T}$ acting on $\boldsymbol{x}^{'}$, where the matrix $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \begin{pmatrix} \ddots & & & & \\ & h_0^{'}(3L-3) & h_0^{'}(3L-2) & \cdots & h_0^{'}(0) \\ & h_1^{'}(3L-3) & h_1^{'}(3L-2) & \cdots & h_1^{'}(0) \\ & 0 & 0 & \cdots & h_1(0) \\ & 0 & h_0^{'}(3L-3) & \cdots & h_0^{'}(1) \\ & 0 & h_1^{'}(3L-3) & \cdots & h_1^{'}(1) \\ & 0 & 0 & \cdots & h_1(1) \\ & & & & & \ddots \end{pmatrix}$$

We concentrate on reconstruction of erasures in the subband components $x_0(n)$ and $x_1(n)$. If a component in these subbands is lost, we loose a value $\langle \boldsymbol{x}^{'}, \boldsymbol{r}_i \rangle$, where $\boldsymbol{r}_i$ is one of the rows of $\boldsymbol{T}$ corresponing to $h_0^{'}$ or $h_1^{'}$. It is possible to reconstruct that component from the received components if and only if $\boldsymbol{r}_i$ can be expressed as a linear combination of other rows of $\boldsymbol{T}$, say $\boldsymbol{r}_i = \sum_{n_k \neq i} a_k \boldsymbol{r}_{n_k}$, such that, $\langle \boldsymbol{x}^{'}, \boldsymbol{r}_{n_k} \rangle$ is not lost for all k in the summation under consideration. In the following sequel, instead of writing 'symbol

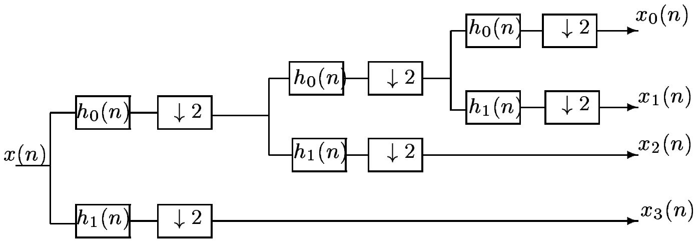

Fig. 1. Tree-structured critically decimated filter bank.

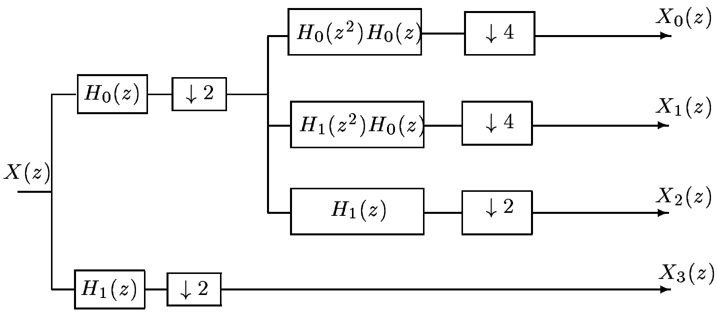

Fig. 2. z-domain representation of the tree-structured filter bank

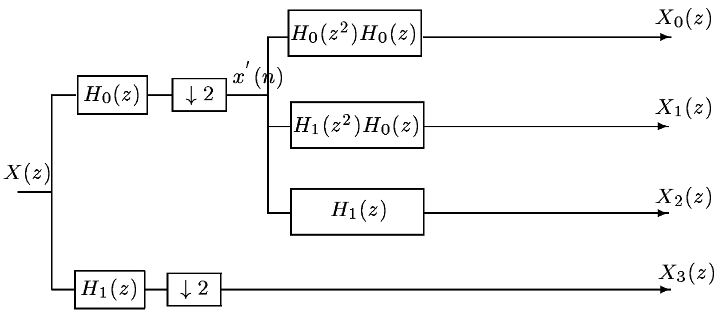

Fig. 3. z-domain representation of the TSOFB

corresponding to inner product of $\boldsymbol{x}$ with a particular row of $\boldsymbol{T}$', we will write 'value corresponding to that particular row'.

3. ANALYSIS OF CONDITIONS FOR PERFECT RECONSTRUCTION

In this section, we analyse conditions under which it is possible to reconstruct erasures in subband components $x_0(n)$ and $x_1(n)$. Observe that the rows of $\boldsymbol{T}$ are right shifts or left shifts of basically three rows of $\boldsymbol{T}$ corresponding to the time-reversed filters $h_0^{'}(n)$, $h_1^{'}(n)$ and $h_1(n)$. We first focus on expressing $h_0^{'}(-n+k)$ (or $h_1^{'}(-n+k)$, for some integer k) as a linear combination of integer shifts of $h_0^{'}(-n)$, $h_1^{'}(-n)$. Note that if the z-transform of the zero$-th$ and first row of $\boldsymbol{T}$ are $H_0^{'}(z)$ and $H_1^{'}(z)$ respectively, then the z-transform of the $3k-th$ and $3k+1-th$ row of $\boldsymbol{T}$ are $z^k H_0^{'}(z)$ and $z^k H_1^{'}(z)$ respectively, for all integer k (We abuse the term z-transform, since we are actually considering the z-transform of the time reversed and shifted versions). If some row can be written as a finite linear combination of other rows, then its z-transform can be written as a corresponding linear combination of the z-transforms. If the zero-th row can be written as a finite linear combination of rows with indices $\{3k_i\}_{i=1}^{M}$ and rows with indices

$\{3n_j+1\}_{j=1}^{N}$, then its z-transform can be written as

$$H_0^{'}(z) = \sum_{i=1}^{M} a_i z^{k_i} H_0^{'}(z) + \sum_{j=1}^{N} b_j z^{n_j} H_1^{'}(z) \quad (2)$$

$$\Longrightarrow \frac{H_0^{'}(z)}{H_1^{'}(z)} = \frac{\sum_{j=1}^{N} b_j z^{n_j}}{1-\sum_{i=1}^{M} a_i z^{k_i}}. \quad (3)$$

Let the value corresponding to the row $H_0(z^2)H_0(z)$ be erased. Let the values corresponding to rows $\{z^{k_i}H_0(z^2)H_0(z)\}_i$, $\{z^{n_j}H_1(z^2)H_0(z)\}_j$ be not erased. If there exists scalars a_i, b_j such that

$$H_0(z^2)H_0(z) = \sum_i a_i z^{k_i} H_0(z^2)H_0(z) + \sum_j b_j z^{n_j} H_1(z^2)H_0(z), \quad (4)$$

then reconstruction of that erased value is possible.

Let the value corresponding to row $H_0(z^2)H_0(z)$ be erased. Let the values corresponding to rows $\{z^{k_i}H_0(z^2)H_0(z)\}_i$, $\{z^{n_j}H_1(z^2)H_0(z)\}_j$, $\{z^{p_m}H_1(z)\}_m$ be not erased.

If there exists scalars a_i, b_j and c_m such that

$$H_0(z^2)H_0(z) = \sum_i a_i z^{k_i} H_0(z^2)H_0(z) + \sum_j b_j z^{n_j} H_1(z^2)H_0(z) + H_1(z)\sum_m c_m z^{p_m}, \quad (5)$$

then reconstruction of that erased value is possible. To know if reconstruction is possible, using (5), we get,

$$\left(1-\sum_i a_i z^{k_i}\right) H_0(z^2)H_0(z) = \sum_j b_j z^{n_j} H_1(z^2)H_0(z) + H_1(z)\sum_m c_m z^{p_m} \quad (6)$$

$$\Longrightarrow H_0(z)\left[H_0(z^2)\left(1-\sum_i a_i z^{k_i}\right) - H_1(z^2)\left(\sum_j b_j z^{n_j}\right)\right] = H_1(z)\left[\sum_m c_m z^{p_m}\right].$$

Since $H_0(z)$ and $H_1(z)$ are relatively prime [2],

$$H_0(z^2)\left(1-\sum_i a_i z^{k_i}\right) - H_1(z^2)\left(\sum_j b_j z^{n_j}\right) = H_1(z)K(z), \quad (7)$$

$$\sum_m c_m z^{p_m} = H_0(z)K(z), \quad (8)$$

for some polynomial $K(z)$. If there exists such a polynomial $K(z)$, then reconstruction is possible. Similar relations can be obtained if the value corresponding to the row $H_1(z^2)H_0(z)$ is erased.

3.1. Bursty Erasure Reconstruction Conditions

In this section, we find bounds on lengths of consecutive erasures in subband components $x_0(n)$ and $x_1(n)$ which can be reconstructed. We first consider expressing $H_0(z^2)H_0(z)$ (or $H_1(z^2)H_0(z)$) as linear combinations of $z^k H_0(z^2)H_0(z)$ and $z^k H_1(z^2)H_0(z)$. From Eqn. (3), we conclude

$$\sum_j b_j z^{n_j} = H_0^{'}(z)K(z), \quad (9)$$

$$1-\sum_j a_i z^{k_i} = H_1^{'}(z)K(z), \quad (10)$$

for some polynomial $K(z)$. The next two results pertain to the erasure burst length bounds for reconstruction of $x_0(n_0)$ or $x_1(n_0)$, for some integer n_0, only from received $x_0(n)$ and $x_1(n)$ (so n can take only selected values).

Theorem 1. For the tree-structured oversampled filter bank shown in Fig. 3, with analysis filters $h_0(n)$ and $h_1(n)$ both of length L, a sufficient condition to reconstruct a burst of $6L-5$ erasures is that the $6L-5$ symbols following and preceding this erasure burst have no erasures.

Proof. If there exists a polynomial $K(z)$ which satisfies Eqn. (9) and (10), for the given burst length, then reconstruction of the bursty erasures is possible. If we define the matrix E as

$$\begin{pmatrix} \ddots & h_1^{'}(0) & 0 & \cdots & 0 & 0 \\ & h_1^{'}(1) & h_1^{'}(0) & \cdots & 0 & 0 \\ & h_0^{'}(1) & h_0^{'}(0) & \cdots & 0 & 0 \\ & & & \ddots & & \\ & h_0^{'}(3L-3) & h_0^{'}(3L-2) & \cdots & h_0^{'}(0) & 0 \\ & 0 & h_1^{'}(3L-3) & \cdots & h_1^{'}(1) & h_1^{'}(0) \\ & 0 & h_0^{'}(3L-3) & \cdots & h_0^{'}(1) & h_0^{'}(0) \end{pmatrix},$$

then, for a erasure burst of length $6L-5$, Eqn. (9) and (10) can be written in matrix form as

$$E\begin{pmatrix} \vdots \\ k_2 \\ k_1 \\ k_0 \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

It can be shown that the $6L-5\times 6L-5$ matrix E is invertible and hence the existence of $K(z)$. Further, from the degree of $K(z)$, we conclude that we need $6L-5$ symbols following and preceding the erasure burst for reconstruction.

Due to space limitations, we only state the next theorem, which can be proved on similar lines as in [6].

Theorem 2. It is not possible to reconstruct a burst of erasures of length $6L-3$ and it is possible to reconstruct a burst of erasures of length $6L-4$ if and only if this burst starts with a $x_1(n)$ for some integer n.

4. RECONSTRUCTION ALGORITHM

The subband components are quantized before transmission. On receiving the data with erasures, we first check if it is possible to reconstruct the erasures in the components $x_0(n)$ and $x_1(n)$ using only the received values corresponding to subband components $x_0(n)$ and $x_1(n)$. To check if this is possible, Theorem 1 and 2 can be used.

For still images and audio data, different subband components are alloted different bit budgets; the low frequency components are quantized using finer quantizers than the high frequency components. Quantization leads to error in the reconstructed signal even if there are no erasures. In the event of erasures, the linear combinations used for reconstruction determine the reconstructed mean square error. Since it is difficult to compute the mean square error analytically, we performed simulations to compute the same.

5. SIMULATION RESULTS

Simulations were performed using an orthogonal filter bank with filters of length $L = 16$. We use a simple image coding system wherein the image is decomposed into 7 subbands using a 2-channel wavelet transform. The rows of the lowest frequency subband component are then input to the oversampled branch of the OFB. So basically the rows act as the signal $x^{'}(n)$ in Fig. 3. Note that this will lead to a rate $R = \frac{1}{3}$ code operating on the lowest frequency subband. In order to perform a fair comparison to the scheme in [6], which uses a rate $\frac{1}{2}$ code, we allocate different bit budgets to the oversampled components, so that the output data rates are the same for both the cases. The reconstruction PSNR values for different burst lengths with quantization of the subband components are tabulated in Table 1 for both, the TSOFB and the OFB. The reconstructed images are shown in Fig. 4. The simulations for the TSOFB are performed using only the linear combinations from $z^k H_0(z^2) H_0(z)$ and $z^k H_1(z^2) H_0(z)$. We notice a significant improvement in the PSNR values for the TSOFB as compared to the OFB.

# of Erasures	Bursty TSOFB	Non-Bursty TSOFB	Bursty OFB	Non-Bursty OFB
0	33.8449	33.8449	33.8406	33.8406
4	33.2995	33.6125	31.3989	31.9125
8	32.1326	33.0231	17.1312	29.0241
12	18.8148	30.2822	-	-

Table 1. PSNR Values for Lena image.

6. CONCLUSIONS

We generalized the erasure reconstruction methods from 2-channel OFB proposed in [6] to tree-structured OFB. Since the filter lengths are larger, the erasure reconstruction capabilities of these filter banks is also better. In the case of quantization, we observe a significant improvement in the reconstruction mean square error values for the TSOFB as compared to block codes and OFB.

Fig. 4. Reconstructed Images- With no erasures (Top Left), Burst of 4 erasures (Top Right), Burst of 8 erasures (Bottom Left), Burst of 12 erasures (Bottom Right).

7. REFERENCES

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